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Local existence and uniqueness of solutions to approximate systems of 1D tumor invasion model

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1. Introduction

ABSTRACT

In the present paper, we propose a modified tumor invasion model which was originally proposed in Chaplain and Anderson (2003) [1]. And we show the local existence and uniqueness of solutions to approximate systems of the 1D modified tumor invasion model. Especially, we introduce a new function and show that our system is equivalent to the nonlinear second-order PDE, which should be reformulated by the new function. Roughly speaking, our system can be rewritten into only one second-order PDE and this fact is quite essential to show the local existence of solutions to the approximate systems.

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(1.4)

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In [1] Chaplain and Anderson proposed the following PDEs-ODE system (S) := $\{(1.1)-(1.4)\}$ to model a tumor invasion phenomenon:

$n_t = \nabla \cdot (\kappa_n(f, m) \nabla n) - \nabla \cdot (n\chi(f) \nabla f) + F_1(n, f, m),$	(1.1)
$f_t = -F_2(f, m),$	(1.2)

 $m_t = \kappa_m \Delta m + g(n, m) - h(n, m, f) - k(m, w),$ (1.3)

 $w_t = \kappa_w \Delta w + l(m, f) - k(m, w) - \varepsilon_w w,$

where the unknown functions *n*, *f*, *m* and *w* represent the concentrations of the tumor cells, the ECM (extracellular matrix), the active MDEs (matrix degrading enzymes) and the endogenous inhibitors, respectively.

The first equation (1.1) describes the kinetics of the tumor cells. In this equation, its flux is given by $-\kappa_n(f, m)\nabla n + n\chi(f)\nabla f$. The former $-\kappa_n(f, m)\nabla n$ represents the random motility of the tumor cells. And a non-negative function $\kappa_n(f, m)$ of f and m represents a chemokinetic response to the ECM and the active MDEs. Roughly speaking, the larger the concentration of the ECM or the active MDEs, the higher the random motility of the tumor cells. The latter $n\chi(f)\nabla f$ describes the haptotactic flux and $\hat{\chi}(f)$ is the haptotactic sensitivity of the tumor cells to the ECM, where $\hat{\chi}$ is the primitive of χ . Moreover, a non-negative function F_1 describes the proliferation of the tumor cells.

The second equation (1.2) describes the kinetics of the ECM. Actually, the ECM is degraded by the biochemical reaction between the ECM and the active MDEs. And its degradation process is described by ODE because this phenomenon is modelled in the meso-scale. By a non-negative function $F_2(f, m)$, we describe the decay rate of the ECM as the result of the biochemical reaction between the ECM and the active MDEs.

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The third equation (1.3) describes the kinetics of the active MDEs. A positive constant κ_m describes the diffusion coefficient of the active MDEs. A positive function g(n, m) describes the production of the active MDEs by the tumor cells and themselves. And positive functions h(n, m, f) and k(m, w) describe the natural decay of the active MDEs, which depends upon the concentrations of the tumor cells and the ECM, and the neutralisation of the active MDEs by the biochemical reaction between the active MDEs and the endogenous inhibitors, respectively.

The fourth equation (1.4) describes the kinetics of the endogenous inhibitors. Positive constants κ_w , ε_w and a positive function l(m, f) describe the diffusion coefficient of the endogenous inhibitors, the natural decay rate and the production by the ECM as a response to the active MDEs of the endogenous inhibitors, respectively.

Now, we propose a modified system of (S). For this, we suppose that the following conditions are satisfied:

- (1) There does not exist any endogenous inhibitors. Hence, we do not consider (1.4).
- (2) The degradation of the ECM occurs when they contact with the tumor cells and therefore we omit the equation for the active MDEs. Roughly speaking, the tumor cells have an influence on the degradation of the ECM directly and we almost identify the concentration of the tumor cells with that of the active MDEs. So, we do not have to consider (1.3). As a result, we drop out the variable *m* of the function F_1 in (1.1) and replace that of F_2 in (1.2) by *n*, namely, $F_1(n, f, m)$ and $F_2(f, m)$ are replaced by $F_1(n, f)$ and $F_2(f, n)$, respectively.
- (3) The coefficient of the random motility is given by a function of space and time, not of the concentrations of the ECM and the active MDEs like $\kappa_n(f, m)$ in (1.1). The reason why it is a function of space and time will be explained below. Throughout this paper we denote it by p = p(x, t) instead of $\kappa_n(f, m)$.
- (4) The proliferation F_1 of the tumor cells in (1.1) is given by a function of space, time and the concentrations of the tumor cells as well as the ECM. The reason why it depends upon space and time will be also explained below. Moreover, we take the apoptosis of the tumor cells into consideration. So, the nonlinear function in (1.1), denoted by F, is expressed by the difference between non-negative functions F_1 (a proliferation of the tumor cells) and F_a (an apoptosis of the tumor cells), i.e., $F = F_1 - F_a$. As a result, we do not have to assume the non-negativeness of F throughout this paper.
- (5) The haptotactic coefficient χ depends upon the concentration of the ECM. In [2], the following functions are reported as the typical examples of χ :

$$\chi(r) = -\frac{\chi_0}{r^2}, \quad \forall r \in (0, +\infty)$$

and

$$\chi(r) = -\frac{\chi_0 K}{(r+K)^3}, \quad \forall r \in [0, +\infty)$$

which are called the logarithmic law and the receptor law, respectively, where χ_0 and K are given positive constants.

(6) The decay of the ECM is directly proportional to the product of the concentrations of the tumor cells and the ECM. We denote by δ its proportion constant, which is positive. Here, you note that the condition (2) above is satisfied.

Under the above setting, we derive the following haptotaxis-degenerate system (P) := $\{(1.5), (1.6)\}$ as a modified tumor invasion model of (S):

$$n_t = \nabla \cdot (p(x, t)\nabla n) - \nabla \cdot (n\chi(f)\nabla f) + F(x, t, n, f),$$

$$f_t = -\delta nf.$$
(1.5)
(1.6)

$$= -\delta n f. \tag{1.6}$$

In below, we explain why a coefficient p = p(x, t) of the random motility of the tumor cells and a function F = F(x, t, n, f)depend upon space and time.

At first, we consider a function F. Recently, it is pointed out that heat shock proteins have influences on the apoptosis of the tumor cells and their dynamics are controlled by a stress of temperature, for example, in [3–6]. In order to take such influences of heat shock proteins into consideration, we assume that a nonlinear function F is a function of space, time and the concentrations of the tumor cells as well as the ECM. But we suppose that the tumor cells and the ECM do not have any influences on the dynamics of heat shock proteins.

Next, we consider a coefficient p. In (S), it is a function of the concentrations of the ECM and the active MDEs. In the process to derive (P), we identify the dynamics of the concentration of the tumor cells with that of the active MDEs. Hence, it must be a function of the concentrations of the tumor cells and the ECM. But, in (P) we suppose that it depends upon distributions of heat shock proteins. So, we give a coefficient of the random motility of the tumor cells by a function of space and time.

Throughout this paper, we impose the following mathematical assumptions to the prescribed data p, χ , F, n_0 and f_0 . In below, let T be any positive and finite time and $\Omega := (-L, L)$ for some positive and finite constant L, which contains all tumor cells.

(A1) *p* is a non-negative and bounded function on $\overline{Q}_T := [-L, L] \times [0, T]$, that is, there exists a positive constant c_1 such that

$$0 \le p(x, t) \le c_1$$
, a.a. $(x, t) \in \overline{Q}_T$.

 $\chi(r) + r\chi(r) \le c_2, \quad \forall r \in \mathbf{R}_+$

and

$$|r_1\chi(r_1) - r_2\chi(r_2)| \le c_3|r_1 - r_2|, \quad \forall r_1, r_2 \in \mathbf{R}_+$$

(A3) *F* is a continuous function from $\overline{Q}_T \times \mathbf{R} \times \mathbf{R}_+$ into **R**. And there exist positive constants c_i (i = 4, 5) such that

 $|F(x, t, 0, 1)| \le c_4, \quad \forall (x, t) \in \overline{Q}_T$

and

$$|F(x, t, r_1, \exp(r_2)) - F(x, t, \tilde{r}_1, \exp(\tilde{r}_2))| \le c_5(|r_1 - \tilde{r}_1| + |r_2 - \tilde{r}_2|)$$

 $\forall (x, t) \in \overline{Q}_T, \forall r_1, \tilde{r}_1 \in \mathbf{R}, \forall r_2, \tilde{r}_2 \in \mathbf{R}_+ (i = 1, 2).$

(A4) $n_0 \in H^1(-L, L)$ with $(n_0)_x(\pm L) = 0$. (A5) $f_0 \in H^1(-L, L)$ with $(f_0)_x(\pm L) = 0$. And there exists a positive constant c_6 such that

 $f_0(x) \ge c_6, \quad \forall x \in [-L, L].$

From the above conditions, $\ln f_0$ is also in $H^1(-L, L)$ with $(\ln f_0)_x(\pm L) = 0$.

Before giving our main theorems of this paper, we state some known mathematical results. Actually, there are a lot of papers treating the following chemotaxis-parabolic PDE system, which is sometimes called the Keller–Segel model:

$$n_t = \kappa_n \Delta n - \nabla \cdot (n\chi(f)\nabla f),$$

$$f_t = \kappa_f \Delta f - F_2(f, n),$$

where κ_n and κ_f are positive constants. For example, in [7] Horstmann gives a survey of the mathematical results for the Keller–Segel model, so, we refer the references to it.

In the present paper we consider the case that $\kappa_f = 0$. In this case the Keller–Segel model becomes a chemotaxisdegenerate system, which is sometimes called the angiogenesis model. For this model, in [8] Friedman and Tello showed the global existence and uniqueness of classical solutions by imposing some suitable assumption to the nonlinear term $F_2(f, n)$. Moreover, in [9] Fontelos, Friedman and Hu considered 1D model for the case $F_2(t, f, n) = a_1nf - a_2(t)f$, where a_1 is a positive constant and a_2 is a positive function on \mathbf{R}_+ . They showed the global existence and uniqueness of classical solutions. Besides, they obtained some results concerned with the steady state problem. Recently, in [10,11] Corrias, Perthame and Zaag considered the higher dimensional models for the case $F_2(f, n) = -nf^m$, where $m \ge 1$, and succeeded in showing the global existence of weak solutions. Especially, they have already succeeded in deriving the global boundedness of solutions with respect to time in some suitable function space for the 2D case.

But, the mathematical treatments, which are developed in the above papers, essentially and strongly depend upon the fact that the coefficient of the random motility κ_n is a positive constant. As a result, they cannot be applied directly for the case κ_n is not a positive constant. Actually, in our model it is given by a non-negative function of space and time, which is denoted by p throughout this paper. Moreover, since p is non-negative and may be not continuous in our setting, the random motility of tumor cells is allowed to be degenerate and discontinuous at some positions and times. From this fact, it is quite difficult to treat our system mathematically. So, we add the terms $\kappa \Delta n_t$ and $-\varepsilon \Delta \ln f$ in the right-hand side of (1.5) to approximate (P).

Finally, in the present paper we consider the following approximate 1D haptotaxis-degenerate system, which is denoted by $(AP) := \{(1.7)-(1.10)\}$ throughout this paper:

$$n_t = \{\kappa n_{tx} + p(x, t)n_x - n\chi(f)f_x - \varepsilon(\ln f)_x\}_x + F(x, t, n, f) \quad \text{a.a. in } Q_T,$$

$$(1.7)$$

$$f_t = -\delta n f \quad \text{a.a. in } Q_T, \tag{1.8}$$

$$r_t (+L, t) = f_t (+L, t) = 0 \quad \text{a.a. } t \in (0, T) \tag{1.9}$$

$$n_x(\pm L, t) = f_x(\pm L, t) = 0$$
 a.a. $t \in (0, T),$ (1.9)

$$n(x,0) = n_0(x), \quad f(x,0) = f_0(x) \quad \text{a.a. } x \in (-L,L), \tag{1.10}$$

 κ and ε are positive constants. Throughout this paper, for simplicity we denote $L^2(-L, L)$ and $H^1(-L, L)$ by H and V, respectively. We note that V is continuously imbedded in C[-L, L], i.e., there exists a positive constant c_7 such that

$$\|z\|_{C[-L,L]} \le c_7 \|z\|_V, \quad \forall z \in V,$$
(1.11)

which is a key inequality throughout all arguments in this paper.

Now, we are in a position to give our main theorems in this paper.

Theorem 1. Assume that (A1)–(A5) hold. Then, there exists $T_0 \in (0, T]$ such that (AP) has at least one solution [n, f] on $[0, T_0]$ satisfying the following properties:

(P1) $n \in W^{1,\infty}(0, T_0; V) (\subset C([0, T_0]; H)).$

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- (P2) $\ln f \in W^{2,\infty}(0, T_0; V) (\subset C^1([0, T_0]; H)).$
- (P3) For any $z \in V$ and a.a. $t \in (0, T_0)$ the following equality holds:

$$(n'(t), z)_{H} + \kappa (n'_{x}(t), z_{x})_{H} + \int_{-L}^{L} p(x, t) n_{x}(x, t) z_{x}(x) dx - \varepsilon ((\ln f)_{x}(t), z_{x})_{H} - \int_{-L}^{L} n(x, t) \chi (f(x, t)) f_{x}(x, t) z_{x}(x) dx = (F(t, n(t), f(t)), z)_{H},$$
(1.12)

where "/" implies the derivative with respect to the variable t throughout this paper.

(P4) For a.a. $(x, t) \in Q_{T_0}$ the following equality holds:

$$f'(x,t) = -\delta n(x,t) f(x,t).$$
(1.13)

(P5) $u(0) = u_0$ and $f(0) = f_0$.

Theorem 2. Assume that (A1)–(A5) hold. Then, (AP) has at most one solution [n, f] on $[0, T_0]$, where T_0 is the same time as in *Theorem* 1.

From Theorems 1 and 2, (AP) has one and only one time-local solution [n, f] on $[0, T_0]$.

2. Equivalent evolution equation to (AP)

Let n_0 and f_0 be the same functions as in (A4) and (A5), respectively. Moreover, for each function ℓ on \overline{Q}_T we define a function $\tilde{\ell}$ by

$$\tilde{\ell}(x,t) := f_0(x) \exp\left(-\delta t n_0(x) - \delta \ell(x,t)\right), \quad \forall (x,t) \in \overline{Q}_T.$$

And we consider the following second-order PDE denoted by $(E) := \{(2.1)-(2.3)\}$:

$$u''(x,t) = q_x(x,t,u(x,t)) + F\left(x,t,u'(x,t) + n_0(x),\tilde{u}(x,t)\right) \quad \text{a.a.} \ (x,t) \in Q_T,$$
(2.1)

$$u_x(\pm L, t) = 0$$
 a.a. $t \in (0, T),$ (2.2)

$$u(x, 0) = u'(x, 0) = 0$$
 a.a. $x \in (-L, L),$ (2.3)

where

$$\begin{aligned} q(x, t, u(x, t)) &= \kappa u_x''(x, t) + p(x, t)u_x'(x, t) + \varepsilon \delta u_x(x, t) + \delta \left\{ u'(x, t) + n_0(x) \right\} \tilde{u}(x, t)\chi \left(\tilde{u}(x, t) \right) u_x(x, t) \\ &+ \left[p(x, t) + \delta t \left\{ u'(x, t) + n_0(x) \right\} \tilde{u}(x, t)\chi \left(\tilde{u}(x, t) \right) + \varepsilon \delta t \right] (n_0)_x(x) \\ &- \left[\left\{ u'(x, t) + n_0(x) \right\} \tilde{u}(x, t)\chi \left(\tilde{u}(x, t) \right) + \varepsilon \right] (\ln f_0)_x(x). \end{aligned}$$

It is easily seen from (A4), (A5) with (2.2) that the following boundary condition holds:

 $q(\pm L, t, u(\pm L, t)) = 0$ a.a. $t \in (0, T)$.

First of all, we give the definition of solutions to (E) below.

Definition 2.1. The function $u : \overline{Q}_T \longrightarrow \mathbf{R}$ is called a solution to (E) on [0, *T*] if and only if the following properties are satisfied:

(E1) $u \in W^{2,\infty}(0,T;V) (\subset C^1([0,T];H) \cap C(\overline{Q}_T)).$ (E2) For any $z \in V$ and a.a. $t \in (0,T)$ the following equality holds:

$$(u''(t), z)_{H} + (q(t, u(t)), z_{x})_{H} = \left(F\left(t, u'(t) + n_{0}, \tilde{u}(t)\right), z\right)_{H}.$$
(2.4)

(E3) u(0) = u'(0) = 0 in *H*.

Then, we see that the following propositions hold.

Proposition 2.1. Let [n, f] be a solution to (AP) on [0, T]. And we define a function $u : \overline{Q}_T \longrightarrow \mathbf{R}$ by

$$u(x,t) := -n_0(x)t - \frac{1}{\delta} \ln \frac{f(x,t)}{f_0(x)}, \quad \forall (x,t) \in \overline{Q}_T.$$

Then, u is a solution to (E).

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Proof. By the standard calculation, it is easily seen that the following equalities are satisfied for a.a. $(x, t) \in \overline{Q}_T$:

$$f(x, t) = f_0(x) \exp(-\delta t n_0(x) - \delta u(x, t)) = \tilde{u}(x, t),$$

$$n(x, t) = u'(x, t) + n_0(x),$$

$$n_x(x, t) = u'_x(x, t) + (n_0)_x(x),$$

$$n'(x, t) = u''(x, t),$$

$$n'_x(x, t) = u''_x(x, t),$$

$$f_x(x, t) = \tilde{u}(x, t) \{-\delta u_x(x, t) - \delta t(n_0)_x(x) + (\ln f_0)_x(x)\},$$

$$(\ln f)_x(x, t) = -\delta u_x(x, t) - \delta t(n_0)_x(x) + (\ln f_0)_x(x).$$

We substitute the above equalities into (1.12). Then, we see that *u* satisfies (2.4).

Moreover, it is clear that u satisfies all regularities and the initial conditions, which are required in (E1) and (E3) in Definition 2.1, respectively.

Hence, we see that *u* is a solution to (E) on [0, T].

Proposition 2.2. Let u be a solution to (E) on [0, T]. Then, we define functions n and f on \overline{Q}_T by

$$n(x,t) := u'(x,t) + n_0(x)$$

and

$$f(x,t) := \tilde{u}(x,t) = f_0(x) \exp\left(-\delta t n_0(x) - \delta u(x,t)\right),$$

respectively. Then, a pair [n, f] is a solution to (AP) on [0, T].

Proof. It is clear that *u* and *f* satisfy the regularities and the initial conditions, which are required in (P1), (P2) and (P5) in Theorem 1. By the standard calculation, we see that the following equalities are satisfied for a.a. $(x, t) \in \overline{Q}_T$:

$$u(x,t) = -tn_0(x) - \frac{1}{\delta} \ln \frac{f(x,t)}{f_0(x)},$$

$$u_x(x,t) = -t(n_0)_x(x) - \frac{1}{\delta} \{(\ln f)_x(x,t) - (\ln f_0)_x(x)\},$$

$$u'_x(x,t) = n_x(x,t) - (n_0)_x(x),$$

$$u''(x,t) = n'(x,t),$$

$$u''_x(x,t) = n'_x(x,t).$$

We substitute the above equalities into (2.4). Then, we see that (1.12) holds. Moreover, we see that the following equality holds for a.a. $(x, t) \in Q_T$:

$$f'(x, t) = -\delta \{ n_0(x) + u'(x, t) \} f_0(x) \exp(-\delta t n_0(x) - \delta u(x, t))$$

= $-\delta n(x, t) f(x, t),$

that is, (1.13) is satisfied.

Hence, we see that a pair [n, f] is a solution to (AP) on [0, T].

From Propositions 2.1 and 2.2, in order to show the local existence of solutions to (AP), it is enough to show that of (E). Actually, in Section 4 we show the local existence of solutions to (E) instead of (AP).

3. Auxiliary problem for (E)

Let v be any given function in $C^1([0, T]; V)$. And we consider the following auxiliary problem (AE)_v = {(3.1)–(3.3)}:

$$u''(x,t) = \tilde{q}_x(x,t,u(x,t),v(x,t)) + F\left(x,t,u'(x,t) + n_0(x),\tilde{u}(x,t)\right) \quad \text{a.a.} (x,t) \in Q_T,$$
(3.1)

$$u_x(\pm L, t) = 0$$
 a.a. $t \in (0, T),$ (3.2)

$$u(x, 0) = u'(x, 0) = 0$$
 a.a. $x \in (-L, L),$ (3.3)

where

$$\begin{split} \tilde{q}(x,t,u(x,t),v(x,t)) &= \kappa u_x''(x,t) + p(x,t)u_x'(x,t) + \varepsilon \delta u_x(x,t) + \delta \left\{ v'(x,t) + n_0(x) \right\} \tilde{v}(x,t) \chi \left(\tilde{v}(x,t) \right) u_x(x,t) \\ &+ \left[p(x,t) + \delta t \left\{ v'(x,t) + n_0(x) \right\} \tilde{v}(x,t) \chi \left(\tilde{v}(x,t) \right) + \varepsilon \delta t \right] (n_0)_x(x) \\ &- \left[\left\{ v'(x,t) + n_0(x) \right\} \tilde{v}(x,t) \chi \left(\tilde{v}(x,t) \right) + \varepsilon \right] (\ln f_0)_x(x). \end{split}$$
(3.4)

This section is devoted to showing the following proposition.

Proposition 3.1. Assume that (A1)–(A5) hold. Then, for each $v \in C^1([0, T]; V)$ (AE)_v has one and only one solution u_v satisfying the following properties:

- (1) $u_v \in W^{2,\infty}(0,T;V) \cap C^1([0,T];V).$
- (2) For any $z \in V$ and a.a. $t \in (0, T)$ the following equality holds:

$$(u_{v}''(t), z)_{H} + \left(\tilde{q}(t, u_{v}(t), v(t)), z_{x}\right)_{H} = \left(F\left(t, u_{v}'(t) + n_{0}, \tilde{u_{v}}(t)\right), z\right)_{H}.$$
(3.5)

(3) $u_v(0) = u'_v(0) = 0$ in *H*.

Moreover, there exists a continuous function $K_1 : \mathbf{R}^4_+ \longrightarrow \mathbf{R}_+$ such that

$$\|u_{v}\|_{C^{1}([0,T];V)} + \|u_{v}''\|_{L^{\infty}(0,T;V)} \le K_{1}(T, \|u_{0}\|_{V}, \|f_{0}\|_{V}, \|v\|_{C^{1}([0,T];V)}),$$
(3.6)

and for each fixed $r_i \in \mathbf{R}_+$ $(1 \le i \le 3)$ the function $K_1(\cdot, r_1, r_2, r_3)$ is strictly increasing on \mathbf{R} as well as for each bounded sets $B_i \subset \mathbf{R}_+$ $(1 \le i \le 3)$

$$\lim_{T \downarrow 0} \sup_{r_i \in B_i, \ 1 \le i \le 3} K_1(T, r_1, r_2, r_3) = 0.$$
(3.7)

In order to show Proposition 3.1 we use Galerkin method. Actually, let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a base of *V*, which is an orthonormal one of *H*, and for each $m \in \mathbb{N}$ we consider the following Galerkin system denoted by $(G)_m = \{(3.8), (3.9)\}$:

$$(u''_{m}(t), z)_{H} + (q(t, u_{m}(t), v(t)), z_{x})_{H} = (F(t, u'_{m}(t) + n_{0}, \tilde{u_{m}}(t)), z)_{H}, \forall z \in V_{m} := \text{span}\{\varphi_{1}, \varphi_{2}, \dots, \varphi_{m}\}, \forall t \in [0, T],$$

$$u_{m}(x, 0) = u'_{m}(x, 0) = 0 \quad \text{a.a.} \ x \in (-L, L),$$

$$(3.9)$$

$$u_m(x,t) = \sum_{i=1}^m a_i(t)\varphi_i(x), \quad \forall (x,t) \in \overline{Q}_T.$$

By using the standard argument in the theory of ODE, it is easily seen that for each $m \in \mathbf{N}$ (G)_m has a unique solution $\boldsymbol{a} = [a_1, a_2, \dots, a_m]^T \in C^2([0, T]; \mathbf{R}^m)$.

Now, we derive the uniform estimates of the solutions u_m to $(G)_m$ in the following lemmas.

Lemma 3.1. There exist continuous functions $R_i : \mathbf{R}^2_+ \longrightarrow \mathbf{R}_+$ (i = 1, 2) such that

$$\sup_{0 \le t \le T} \|u'_{m}(t)\|_{V}^{2} + \sup_{0 \le t \le T} \|u_{m}(t)\|_{V}^{2} \le \left(1 + \|v\|_{C^{1}([0,T];V)}^{2}\right) R_{1}(\|n_{0}\|_{V}, \|f_{0}\|_{V}) \times \left(\frac{T^{3}}{3} + T\right) \exp\left(T\left(1 + \|v\|_{C^{1}([0,T];V)}\right) R_{2}(\|n_{0}\|_{V}, \|f_{0}\|_{V})\right), \quad \forall m \in \mathbb{N}.$$

$$(3.10)$$

Proof. For simplicity, we skip the index *m*. We substitute z = u'(t) in (3.8). Then, it is easily seen from (A1)–(A5) that there exist constants $C_i > 0$ (i = 1, 2), which are determined by $||n_0||_V$ and $||f_0||_V$, such that the following inequality holds for a.a. $t \in (0, T)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}G_{1}(t) \leq C_{1}\left(1 + \|v\|_{C^{1}([0,T];V)}\right)G_{1}(t) + C_{2}(t^{2}+1)\left(1 + \|v\|_{C^{1}([0,T];V)}^{2}\right)$$
(3.11)

where

 $G_1(t) := \|u'(t)\|_{H}^2 + \kappa \|u'_{x}(t)\|_{H}^2 + \varepsilon \delta \|u(t)\|_{V}^2.$

By applying Gronwall lemma to (3.11), we see that (3.10) holds. \diamond

Lemma 3.2. There exists a constant $R_3 > 0$, which is determined by T, $||n_0||_V$, $||f_0||_V$ and $||v||_{C^1([0,T];V)}$, such that

 $\|u_m''\|_{L^{\infty}(0,T;V)} \leq R_3, \quad \forall m \in \mathbf{N}.$

Proof. By substituting $z = u''_m(t)$ in (3.8) and using Lemma 3.1 with (A1)–(A5) again, we can easily obtain that this lemma holds. \diamond

Now, we are in a position to show Proposition 3.1.

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Proof of Proposition 3.1. First, we show the existence of solutions to $(AE)_v$. From Lemmas 3.1 and 3.2, we can take a subsequence $\{m_k\} \subset \{m\}$ and a function *u* so that

$$u \in W^{2,\infty}(0,T;V) (\subset C^1([0,T];V))$$

and

$$u_{k} := u_{m_{k}} \longrightarrow u \quad \begin{cases} \text{strongly in } C^{1}([0, T]; H), \\ \text{*-weakly in } W^{2,\infty}(0, T; V) \end{cases}$$
(3.12)

as $k \to \infty$.

Moreover, by taking a subsequence of $\{m_k\}$ if it is necessary, which is denoted by the same notation $\{m_k\}$ throughout this proof, we see that

$$u_k(x,t) \longrightarrow u(x,t), \qquad u'_k(x,t) \longrightarrow u'(x,t) \quad \text{a.a.} (x,t) \in Q_T$$

$$(3.13)$$

as $k \to \infty$.

We see from (3.4) and (3.5) that for any $k \in \mathbf{N}$ and $\phi \in C_0^{\infty}(0, T)$ the following equality holds:

$$\begin{split} &\int_{0}^{T} (u_{k}''(t), z)_{H} \phi(t) dt + \kappa \int_{0}^{T} ((u_{k}'')_{x}(t), z_{x})_{H} \phi(t) dt + \int_{0}^{T} \int_{-L}^{L} p(x, t)(u_{k}')_{x}(x, t) z_{x}(x) \phi(t) dx dt \\ &+ \varepsilon \delta \int_{0}^{T} ((u_{k})_{x}(t), z_{x})_{H} \phi(t) dt + \delta \int_{0}^{T} \int_{-L}^{L} \left\{ v'(x, t) + n_{0}(x) \right\} \tilde{v}(x, t) \chi(\tilde{v}(x, t))(u_{k})_{x}(x, t) z_{x}(x) \phi(t) dx dt \\ &= -\int_{0}^{T} \int_{-L}^{L} p(x, t)(n_{0})_{x}(x) z_{x}(x) \phi(t) dx dt + \varepsilon \int_{0}^{T} ((\ln f_{0})_{x} - \delta t(n_{0})_{x}, z_{x})_{H} \phi(t) dt \\ &+ \int_{0}^{T} \int_{-L}^{L} \left\{ v'(x, t) + n_{0}(x) \right\} \tilde{v}(x, t) \chi(\tilde{v}(x, t)) \left\{ (\ln f_{0})_{x}(x) - \delta t(n_{0})_{x}(x) \right\} z_{x}(x) \phi(t) dx dt \\ &+ \int_{0}^{T} \left(F \left(t, u_{k}'(t) + n_{0}, \tilde{u}_{k}(t) \right), z \right)_{H} \phi(t) dt, \quad \forall z \in V_{m_{k}}. \end{split}$$

We substitute $z = \varphi_j$ (j = 1, 2, 3, ...) and take the limit $k \to \infty$ in the above equality. Then, it is easily seen from the convergences in (3.12) and (3.13) that the following equality holds:

$$\int_{0}^{T} (u''(t), \varphi_{j})_{H} \phi(t) dt + \kappa \int_{0}^{T} (u''_{x}(t), (\varphi_{j})_{x})_{H} \phi(t) dt + \int_{0}^{T} \int_{-L}^{L} p(x, t) u'_{x}(x, t) (\varphi_{j})_{x}(x) \phi(t) dx dt + \varepsilon \delta \int_{0}^{T} (u_{x}(t), (\varphi_{j})_{x})_{H} \phi(t) dt + \delta \int_{0}^{T} \int_{-L}^{L} \{v'(x, t) + n_{0}(x)\} \tilde{v}(x, t) \chi(\tilde{v}(x, t)) u_{x}(x, t) (\varphi_{j})_{x}(x) \phi(t) dx dt = -\int_{0}^{T} \int_{-L}^{L} p(x, t) (n_{0})_{x}(x) (\varphi_{j})_{x}(x) \phi(t) dx dt + \varepsilon \int_{0}^{T} ((\ln f_{0})_{x} - \delta t(n_{0})_{x}, (\varphi_{j})_{x})_{H} \phi(t) dt + \int_{0}^{T} \int_{-L}^{L} \{v'(x, t) + n_{0}(x)\} \tilde{v}(x, t) \chi(\tilde{v}(x, t)) (\ln f_{0})_{x}(x) (\varphi_{j})_{x}(x) \phi(t) dx dt - \delta \int_{0}^{T} \int_{-L}^{L} t \{v'(x, t) + n_{0}(x)\} \tilde{v}(x, t) \chi(\tilde{v}(x, t)) (n_{0})_{x}(x) (\varphi_{j})_{x}(x) \phi(t) dx dt + \int_{0}^{T} (F(t, u'(t) + n_{0}, \tilde{u}(t)), \varphi_{j})_{H} \phi(t) dt.$$
(3.14)

So, (3.14) is valid for any $z \in \bigcup_{m=1}^{\infty} V_m$. Since the set $\bigcup_{m=1}^{\infty} V_m$ is dense in *V*, we see that the equality (3.14) holds for any $z \in V$ and $\phi \in C_0^{\infty}(0, T)$. This implies that *u* is a solution to $(AE)_v$.

Moreover, it is clear from Lemmas 3.1 and 3.2 that there exists a function *K*₁ satisfying all properties, which are demanded in Proposition 3.1.

In the rest of this proof, we show the uniqueness of solutions to $(AE)_v$. For this, let u_i (i = 1, 2) be two solutions to $(AE)_v$, and put $\overline{U} := u_1 - u_2$. Then, we see that for any $z \in V$ and a.a. $t \in (0, T)$ the following equality holds:

$$(\bar{U}''(t), z)_{H} + \kappa (\bar{U}''_{x}(t), z_{x})_{H} + \int_{-L}^{L} p(x, t) \bar{U}'_{x}(x, t) z_{x}(x) dx + \delta \int_{-L}^{L} \{ v'(x, t) + n_{0}(x) \} \tilde{v}(x, t) \chi (\tilde{v}(x, t)) \bar{U}_{x}(x, t) z_{x}(x) dx + \varepsilon \delta (\bar{U}_{x}(t), z_{x})_{H} = \left(F \left(t, u_{1}(t) + n_{0}, \tilde{u_{1}}(t) \right) - F \left(t, u_{2}(t) + n_{0}, \tilde{u_{2}}(t) \right), z \right)_{H}.$$
(3.15)

We substitute $z = \overline{U}'(t)$ in (3.15) and use (A1), (A2) to derive

$$\begin{aligned} &\frac{1}{2} \left\{ \|\bar{U}'(t)\|_{H}^{2} + \kappa \|\bar{U}_{x}'(t)\|_{H} + \varepsilon \delta \|\bar{U}_{x}(t)\|_{H}^{2} \right\} \leq \delta c_{2} \int_{-L}^{L} \left(|v'(x,t)| + |n_{0}(x)| \right) |\bar{U}_{x}(x,t)| |\bar{U}_{x}'(x,t)| dx \\ &+ \int_{-L}^{L} \left| F\left(x,t,u_{1}(x,t) + n_{0}(x),\tilde{u}_{1}(x,t) \right) - F\left(x,t,u_{2}(x,t) + n_{0}(x),\tilde{u}_{2}(x,t) \right) \right| |\bar{U}'(x,t)| dx. \end{aligned}$$

It is easily seen from (1.11), (A3) and (A4) that

$$\delta c_2 \int_{-L}^{L} \left(|v'(x,t)| + |n_0(x)| \right) |\bar{U}_x(x,t)| |\bar{U}'_x(x,t)| dx \le C \left(||v'(t)||_V + 1 \right) \left\{ ||\bar{U}_x(t)||_H^2 + ||\bar{U}'_x(t)||_H^2 \right\}$$

and

$$\begin{split} &\int_{-L}^{L} \left| F\left(x,t,u_{1}(x,t)+n_{0}(x),\tilde{u_{1}}(x,t)\right)-F\left(x,t,u_{2}(x,t)+n_{0}(x),\tilde{u_{2}}(x,t)\right) \right| \left| \bar{U}'(x,t) \right| dx \\ &= \int_{-L}^{L} \left| F\left(x,t,u_{1}(x,t)+n_{0}(x),\exp(\ln f_{0}(x)-\delta\{tn_{0}(x)+u_{1}(x,t)\})\right) \right| \\ &- F\left(x,t,u_{2}(x,t)+n_{0}(x),\exp(\ln f_{0}(x)-\delta\{tn_{0}(x)+u_{2}(x,t)\})\right) \left| \left| \bar{U}'(x,t) \right| dx \\ &\leq C\left(\left\| \bar{U}'(t) \right\|_{H}^{2} + \left\| \bar{U}(t) \right\|_{H}^{2} \right). \end{split}$$

Hence, we derive that the following inequality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}G_2(t) \le C\left(\|v\|_{C^1([0,T];V)} + 1\right)G_2(t), \quad \text{a.a. } t \in (0,T),$$
(3.16)

where

$$G_{2}(t) := \|\bar{U}'(t)\|_{H}^{2} + \kappa \|\bar{U}'_{x}(t)\|_{H}^{2} + \varepsilon \delta \|\bar{U}(t)\|_{V}^{2}$$

By applying Gronwall lemma to (3.16), it follows that

 $\|\bar{U}'(t)\|_{H}^{2} + \kappa \|\bar{U}'_{x}(t)\|_{H}^{2} + \varepsilon \delta \|\bar{U}(t)\|_{V}^{2} \leq 0, \quad \forall t \in [0, T].$

This implies that $u_1(t) = u_2(t)$ in *V* for any $t \in [0, T]$. Hence, we see that $(AE)_v$ has at most one solution on [0, T].

4. Proof of Theorem 1

We devote this section to showing Theorem 1. For this, throughout this section we fix any number R > 0. And for each finite T > 0 we consider the closed ball $B_R(T)$ with center 0 and radius R of $C^1([0, T]; V)$, i.e.,

 $B_R(T) := \left\{ v \in C^1([0, T]; V) \mid ||v||_{C^1([0, T]; V)} \le R \right\}.$

From Proposition 3.1, we can define an operator S_T from $B_R(T)$ into $C^1([0, T]; V)$, which assigns $v \in B_R(T)$ to a unique solution $u_v := S_T v$ to $(AE)_v$.

Then, we derive the following lemma.

Lemma 4.1. There exists a finite $T_1 := T_1(R) > 0$ such that

 $S_T B_R(T) \subset B_R(T), \quad \forall T \in (0, T_1],$

i.e., for any $T \in (0, T_1]$ S_T is an operator from $B_R(T)$ into itself.

Proof. From (3.6) in Proposition 3.1, we see that the following inequality holds:

 $\|S_T v\|_{C^1([0,T];V)} \le K_1(T, \|u_0\|_V, \|f_0\|_V, R), \quad \forall v \in B_R(T),$

where K_1 is the same function as in Proposition 3.1.

Since the function $K_1(\cdot, ||u_0||_V, ||f_0||_V, R)$ is a strictly increasing and continuous function on \mathbf{R}_+ satisfying

 $K_1(0, ||u_0||_V, ||f_0||_V, R) = 0, \quad \lim_{T \uparrow \infty} K_1(T, ||u_0||_V, ||f_0||_V, R) = \infty,$

it is easily seen that the algebraic equation $K_1(t, ||u_0||_V, ||f_0||_V, R) = R$ has a unique solution $T_1 := T_1(R)$. Then, it is clear that

 $||S_T v||_{C^1([0,T];V)} \le R, \quad \forall T \in (0, T_1], \ \forall v \in B_R(T),$

that is, $S_T B_R(T) \subset B_R(T)$ for all $T \in (0, T_1]$.

To show the local existence of solutions to (E) it is enough to show Lemma 4.2. Actually, from Lemma 4.2 we can apply Banach fixed point theorem and it is clear that its fixed point is a solution to (E) on $[0, T_2]$, where T_2 is the same time as in Lemma 4.2. So, we show Lemma 4.2 and omit the exact proof of Theorem 1 in this paper.

Lemma 4.2. There exists a finite $T_2 := T_2(R) \in (0, T_1]$ such that

 $\|S_{T_2}v_1 - S_{T_2}v_2\|_{C^1([0,T_2];V)} < \|v_1 - v_2\|_{C^1([0,T_2];V)}, \quad \forall v_1, v_2 \in B_R(T_2),$

that is, S_{T_2} is contractive with respect to the strong topology of $C^1([0, T_2]; V)$, where T_1 is the same time as in Lemma 4.1.

Proof. In this proof, we denote by *C* positive constants, which are determined by $||n_0||_V$, $||f_0||_V$, *R* and constants c_i given in Section 1, but are independent of *T*. Moreover, for simplicity we put $u_i := S_T v_i$ (i = 1, 2), $\overline{U} := u_1 - u_2$ and $\overline{V} := v_1 - v_2$. Then, for each $T \in (0, T_1]$ it follows from Lemma 4.1 that

$$\|u_i\|_{C^1([0,T];V)} \le R, \quad i = 1, 2.$$
(4.1)

We substitute $z := \overline{U}'(t)$ in the following equality:

$$\begin{split} &(\bar{U}''(t), z)_{H} + \kappa(\bar{U}_{x}''(t), z_{x})_{H} + \int_{-L}^{L} p(x, t)\bar{U}_{x}'(x, t)z_{x}(x)dx \\ &+ \delta \int_{-L}^{L} \left\{ v_{1}'(x, t) + n_{0}(x) \right\} \tilde{v}_{1}(x, t)\chi(\tilde{v}_{1}(x, t))(u_{1})_{x}(x, t)z_{x}(x)dx \\ &- \delta \int_{-L}^{L} \left\{ v_{2}'(x, t) + n_{0}(x) \right\} \tilde{v}_{2}(x, t)\chi(\tilde{v}_{2}(x, t))(u_{2})_{x}(x, t)z_{x}(x)dx + \varepsilon\delta(\bar{U}_{x}(t), z_{x})_{H} \\ &= \int_{-L}^{L} \left\{ v_{1}'(x, t) + n_{0}(x) \right\} \tilde{v}_{1}(x, t)\chi(\tilde{v}_{1}(x, t)) \left\{ (\ln f_{0})_{x}(x) - \delta t(n_{0})_{x}(x) \right\} z_{x}(x)dx \\ &- \int_{-L}^{L} \left\{ v_{2}'(x, t) + n_{0}(x) \right\} \tilde{v}_{2}(x, t)\chi(\tilde{v}_{2}(x, t)) \left\{ (\ln f_{0})_{x}(x) - \delta t(n_{0})_{x}(x) \right\} z_{x}(x)dx \\ &+ \int_{-L}^{L} \left\{ F\left(x, t, u_{1}'(x, t) + n_{0}(x), \tilde{u}_{1}(x, t)\right) - F\left(x, t, u_{2}'(x, t) + n_{0}(x), \tilde{u}_{2}(x, t)\right) \right\} z(x)dx. \end{split}$$

First of all, we give the following estimate, which is a key inequality in this proof. We see from (1.11), (A2), (A4) and (A5) that

$$\begin{aligned} |\tilde{v_1}(x,t) - \tilde{v_2}(x,t)| &= f_0(x) |\exp(-\delta\{tn_0(x) + v_1(x,t)\}) - \exp(-\delta\{tn_0(x) + v_2(x,t)\})| \\ &\leq \delta f_0(x) \exp(\delta\{2t|n_0(x)| + |v_1(x,t)| + |v_2(x,t)|\}) |v_1(x,t) - v_2(x,t)| \\ &\leq \delta c_7^2 ||f_0||_V \exp(\delta c_7(2t||n_0||_V + ||v_1(t)||_V + ||v_2(t)||_V)) ||\bar{V}(t)||_V \\ &\leq \delta c_7^2 ||f_0||_V \exp(2\delta c_7(||n_0||_V T + R)) ||\bar{V}(t)||_V, \end{aligned}$$

that is,

 $\|\tilde{v_1} - \tilde{v_2}\|_{L^{\infty}(Q_T)} \le C \exp(C(T+1)) \|\bar{V}\|_{C([0,T];V)}.$

Secondly, it follows from (1.11), (4.1) and (4.2) that the following inequality holds:

$$\begin{split} &-\delta \int_{-L}^{L} \left\{ v_{1}'(x,t)\tilde{v}_{1}(x,t)\chi\left(\tilde{v}_{1}(x,t)\right)\left(u_{1}\right)_{x}(x,t) - v_{2}'(x,t)\tilde{v}_{2}(x,t)\chi\left(\tilde{v}_{2}(x,t)\right)\left(u_{2}\right)_{x}(x,t)\right\} \bar{U}_{x}'(x,t)dx \\ &\leq \delta \int_{-L}^{L} |\bar{V}'(x,t)| |\tilde{v}_{1}(x,t)\chi\left(\tilde{v}_{1}(x,t)\right) ||(u_{1})_{x}(x,t)| |\bar{U}_{x}'(x,t)|dx \\ &+ \delta \int_{-L}^{L} |v_{2}'(x,t)| |\tilde{v}_{1}(x,t)\chi\left(\tilde{v}_{1}(x,t)\right) - \tilde{v}_{2}(x,t)\chi\left(\tilde{v}_{2}(x,t)\right) ||(u_{1})_{x}(x,t)| |\bar{U}_{x}'(x,t)|dx \\ &+ \delta \int_{-L}^{L} |v_{2}'(x,t)| |\tilde{v}_{2}(x,t)\chi\left(\tilde{v}_{2}(x,t)\right) ||\bar{U}_{x}(x,t)| |\bar{U}_{x}'(x,t)|dx \\ &\leq \delta c_{2}c_{7} \|u_{1}(t)\|_{V} \|\bar{V}'(t)\|_{V} \|\bar{U}_{x}'(t)\|_{H} + \delta c_{2}c_{7} \|v_{2}'(t)\|_{V} \|\bar{U}_{x}(t)\|_{H} \|\bar{U}_{x}'(t)\|_{H} \\ &+ \delta c_{3}c_{7} \|v_{2}'(t)\|_{V} \int_{-L}^{L} |\tilde{v}_{1}(x,t) - \tilde{v}_{2}(x,t)| |(u_{1})_{x}(x,t)| |\bar{U}_{x}'(x,t)|dx \\ &\leq \delta c_{2}c_{7} R \left\{ \|\bar{V}'(t)\|_{V} \|\bar{U}_{x}'(t)\|_{H} + \|\bar{U}_{x}(t)\|_{H} \|\bar{U}_{x}'(t)\|_{H} \right\} + \delta c_{3}c_{7} R^{\frac{3}{2}} \|\tilde{v}_{1} - \tilde{v}_{2}\|_{L^{\infty}(Q_{T})} \|\bar{U}_{x}'(t)\|_{H} \\ &\leq C \left\{ \|\bar{U}_{x}(t)\|_{H}^{2} + \|\bar{U}_{x}'(t)\|_{H}^{2} \right\} + C \exp(C(T+1)) \|\bar{V}\|_{C^{1}([0,T];V)}^{2}. \end{split}$$

(4.2)

By repeating the similar argument above, we derive the following inequalities:

•
$$-\delta \int_{-L}^{L} n_0(x) \{ \tilde{v}_1(x,t) \chi(\tilde{v}_1(x,t))(u_1)_x(x,t) - \tilde{v}_2(x,t) \chi(\tilde{v}_2(x,t))(u_2)_x(x,t) \} \bar{U}'_x(x,t) dx$$

 $\leq C \{ \| \bar{U}_x(t) \|_{H}^{2} + \| \bar{U}'_x(t) \|_{H}^{2} \} + C \exp(C(T+1)) \| \bar{V} \|_{C([0,T];V)}^{2},$
• $\int_{-L}^{L} \{ v'_1(x,t) \tilde{v}_1(x,t) \chi(\tilde{v}_1(x,t)) - v'_2(x,t) \tilde{v}_2(x,t) \chi(\tilde{v}_2(x,t)) \} (\ln f_0)_x(x) \bar{U}'_x(x,t) dx$
 $\leq C \| \bar{U}'_x(t) \|_{H}^{2} + C \exp(C(T+1)) \| \bar{V} \|_{C^1([0,T];V)}^{2},$
• $\delta t \int_{-L}^{L} \{ v'_1(x,t) \tilde{v}_1(x,t) \chi(\tilde{v}_1(x,t)) - v'_2(x,t) \tilde{v}_2(x,t) \chi(\tilde{v}_2(x,t)) \} (n_0)_x(x) \bar{U}'_x(x,t) dx$
 $\leq C \| \bar{U}'_x(t) \|_{H}^{2} + Ct^2 \exp(C(T+1)) \| \bar{V} \|_{C^1([0,T];V)}^{2},$
• $\int_{-L}^{L} n_0(x) \{ \tilde{v}_1(x,t) \chi(\tilde{v}_1(x,t)) - \tilde{v}_2(x,t) \chi(\tilde{v}_2(x,t)) \} (\ln f_0)_x(x) \bar{U}'_x(x,t) dx$
 $\leq C \| \bar{U}'_x(t) \|_{H}^{2} + C \exp(C(T+1)) \| \bar{V} \|_{C([0,T];V)}^{2},$

•
$$-\delta t \int_{-L} n_0(x) \{ \tilde{v_1}(x,t) \chi (\tilde{v_1}(x,t)) - \tilde{v_2}(x,t) \chi (\tilde{v_2}(x,t)) \} (n_0)_x(x) \bar{U}'_x(x,t) dx$$

 $\leq C \| \bar{U}'_x(t) \|_{H}^2 + Ct^2 \exp(C(T+1)) \| \bar{V} \|_{C([0,T];V)}^2.$

Thirdly, it follows from (A3) that the following inequality holds:

$$\int_{-L}^{L} \left\{ F\left(x, t, u_{1}'(x, t) + n_{0}(x), \tilde{u_{1}}(x, t)\right) - F\left(x, t, u_{2}'(x, t) + n_{0}(x), \tilde{u_{2}}(x, t)\right) \right\} \bar{U}(x, t) dx$$

$$\leq C \left\{ \|\bar{U}'(t)\|_{H}^{2} + \|\bar{U}(t)\|_{H}^{2} \right\}, \quad \text{a.a. } t \in (0, T).$$

At last, we derive from the above inequalities that the following inequality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t}G_{3}(t) \leq C\left\{G_{3}(t) + (t^{2} + 1)\exp(C(T + 1))\|\bar{V}\|_{C^{1}([0,T];V)}^{2}\right\}, \quad \text{a.a.} \ t \in (0,T),$$

$$(4.3)$$

where

.1

$$G_{3}(t) := \|\bar{U}'(t)\|_{H}^{2} + \kappa \|\bar{U}'_{X}(t)\|_{H}^{2} + \varepsilon \delta \|\bar{U}(t)\|_{V}^{2}.$$

By applying Gronwall lemma to (4.3), we see that there exists a strictly increasing and continuous function R_4 from \mathbf{R}_+ into itself such that

$$\|\bar{U}\|_{C^{1}([0,T];V)} \le R_{4}(T) \|\bar{V}\|_{C^{1}([0,T];V)}, \quad \forall T \in (0,T_{1}]$$

$$(4.4)$$

and

$$R_4(0) = 0, \quad \lim_{T \uparrow \infty} R_4(T) = \infty.$$
 (4.5)

Hence, we put by \tilde{T}_2 a unique solution of the algebraic equation $R_4(T) = 1$ and choose a positive and finite T_2 so that $T_2 < \tilde{T}_2$ and $T_2 \leq T_1$. Then, it is clear from (4.4) and (4.5) that T_2 is a desired one in this lemma, that is, S_{T_2} is a contractive operator from $B_R(T_2)$ into itself with respect to the strong topology of $C^1([0, T_2]; V)$.

5. Proof of Theorem 2

We devote this section to showing Theorem 2. Before giving its proof, we prepare the following lemma, which is a quite standard inequality.

Lemma 5.1. The following inequality holds:

$$|r_1 - r_2| \le (r_1 + r_2) |\ln r_1 - \ln r_2|, \quad \forall r_1, r_2 \in (0, \infty).$$
(5.1)

Proof. By using Taylor expansion theorem and the convexity of the function $-\ln r$, we see

$$r_1 - r_2 \le r_1(\ln r_1 - \ln r_2), \quad \forall r_1, r_2 \in (0, \infty).$$

Hence, we derive

$$r_1 - r_2 \le r_1(\ln r_1 - \ln r_2) \le (r_1 + r_2)|\ln r_1 - \ln r_2|$$

and

$$r_2 - r_1 \le r_2(\ln r_2 - \ln r_1) \le (r_1 + r_2) |\ln r_1 - \ln r_2|.$$

This implies that (5.1) holds. \diamond

Proof of Theorem 2. Throughout this proof, the constants c_i is the same constants as in Section 1 and we denote by *C* positive constants. Let $[n_i, f_i]$ (i = 1, 2) be solutions to (AP) on $[0, T_0]$ and put $N := n_1 - n_2$ and $F := f_1/f_2$ for simplicity. We note that for any $z \in V$ and a.a. $t \in (0, T_0)$ the following equality holds:

$$(N'(t), z)_{H} + \kappa (N'_{x}(t), z_{x})_{H} + \int_{-L}^{L} p(x, t)N_{x}(x, t)z_{x}(x) - \varepsilon ((\ln F)_{x}(t), z_{x})_{H}$$

$$- \int_{-L}^{L} \{n_{1}(x, t)\chi(f_{1}(x, t))(f_{1})_{x}(x, t) - n_{2}(x, t)\chi(f_{2}(x, t))(f_{2})_{x}(x, t)\}z_{x}(x)dx$$

$$= \int_{-L}^{L} \{F(x, t, n_{1}(x, t), f_{1}(x, t)) - F(x, t, n_{2}(x, t), f_{2}(x, t))\}z(x)dx.$$
(5.2)

By substituting z = N(t) in (5.2) and using (A1), we derive the following inequality:

$$\frac{1}{2} \frac{d}{dt} \left\{ \|N(t)\|_{H}^{2} + \kappa \|N_{x}(t)\|_{H}^{2} \right\} - \varepsilon((\ln F)_{x}(t), N_{x}(t))_{H} \\
\leq \int_{-L}^{L} |n_{1}(x, t)\chi(f_{1}(x, t))(f_{1})_{x}(x, t) - n_{2}(x, t)\chi(f_{2}(x, t))(f_{2})_{x}(x, t)||N_{x}(x, t)|dx \\
+ \int_{-L}^{L} |F(x, t, n_{1}(x, t), f_{1}(x, t)) - F(x, t, n_{2}(x, t), f_{2}(x, t))||N(x, t)|dx \\
=: I_{1}(t) + I_{2}(t), \quad \text{a.a. } t \in (0, T_{0}).$$
(5.3)

First of all, we calculate the last term in the left-hand side of (5.3). We see from (1.13) and the regularities of n_i and $\ln f_i$ (cf. (P1) and (P2) in Theorem 1) that

$$(\ln F)_{x}(x, t) = -\delta \int_{0}^{t} N_{x}(x, s) ds, \quad \text{a.a.} (x, t) \in Q_{T_{0}}.$$
 (5.4)

By using (5.4), we see that the following equality holds:

$$-\varepsilon((\ln F)_{X}(t), N_{X}(t))_{H} = \frac{\varepsilon}{2\delta} \frac{\mathrm{d}}{\mathrm{d}t} \|(\ln F)_{X}(t)\|_{H}^{2}, \quad \text{a.a. } t \in (0, T_{0}).$$

$$(5.5)$$

Secondly, we estimate $I_1(t)$. It follows from (A2), (1.11) and Lemma 5.1 that

$$\begin{split} I_{1}(t) &\leq c_{2}c_{7} \left\{ \|(\ln f_{1})_{x}(t)\|_{H} \|N(t)\|_{V} \|N_{x}(t)\|_{H} + \|n_{2}(t)\|_{V} \|(\ln F)_{x}(t)\|_{H} \|N_{x}(t)\|_{H} \right\} \\ &+ c_{3}c_{7} \|n_{2}(t)\|_{V} \int_{-L}^{L} |(\ln f_{2})_{x}(x,t)| \|f_{1}(x,t) - f_{2}(x,t)| |N_{x}(x,t)| dx \\ &\leq c_{2}c_{7} \left[\|\ln f_{1}(t)\|_{V} \|N(t)\|_{V}^{2} + \frac{\|n_{2}(t)\|_{V}}{2} \left\{ \|(\ln F)_{x}(t)\|_{H}^{2} + \|N_{x}(t)\|_{H}^{2} \right\} \right] \\ &+ c_{3}c_{7} \|n_{2}(t)\|_{V} \int_{-L}^{L} \{f_{1}(x,t) + f_{2}(x,t)\} |(\ln f_{2})_{x}(x,t)| |\ln F(x,t)| N_{x}(x,t)| dx \\ &\leq C \left\{ \|n_{2}(t)\|_{V} + \|\ln f_{1}(t)\|_{V} \right\} \left\{ \|N(t)\|_{V}^{2} + \|\ln F(t)\|_{V}^{2} \right\} \\ &+ c_{3}c_{7}^{3} \|n_{2}(t)\|_{V} \left\{ \|f_{1}(t)\|_{V} + \|f_{2}(t)\|_{V} \right\} \|\ln f_{2}(t)\|_{V} \|\ln F(t)\|_{V} \|N(t)\|_{V}, \end{split}$$

hence,

$$I_1(t) \le K_1(t) \left\{ \|N(t)\|_V^2 + \|\ln F(t)\|_V^2 \right\}, \quad \text{a.a. } t \in (0, T_0),$$

(5.6)

where

$$K_1(t) := C \left[\|n_2(t)\|_V + \|\ln f_1(t)\|_V + \|n_2(t)\|_V \|\ln f_2(t)\|_V \left\{ \|f_1(t)\|_V + \|f_2(t)\|_V \right\} \right].$$

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It is easily seen from Theorem 1 that K_1 is a non-negative function of $L^{\infty}(0, T_0)$.

Thirdly, we estimate $I_2(t)$. It is easily seen from (A3) that the following inequality holds:

$$I_2(t) \le C\left\{\|N(t)\|_{H}^{2} + \|\ln F(t)\|_{H}^{2}\right\}, \quad \text{a.a. } t \in (0, T_0).$$
(5.7)

Fourthly, since it follows from (1.13) that

$$(\ln F)'(x, t) = -\delta N(x, t), \quad \text{a.a.} (x, t) \in Q_{T_0},$$

it is easily seen that the following inequality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\ln F(t)\|_{H}^{2} \le \delta \left\{ \|N(t)\|_{H}^{2} + \|\ln F(t)\|_{H}^{2} \right\}, \quad \text{a.a. } t \in (0, T_{0}).$$
(5.8)

At last, it follows from (5.3) and (5.5)–(5.8) that there exists a non-negative function $K_2 \in L^{\infty}(0, T_0)$ such that the following inequality holds:

$$\frac{d}{dt}G_4(t) \le K_2(t)G_4(t), \quad \text{a.a. } t \in (0, T_0),$$
(5.9)

where

$$G_4(t) := \|N(t)\|_H^2 + \kappa \|N_x(t)\|_V^2 + \|\ln F(t)\|_H^2 + \frac{\varepsilon}{\delta} \|(\ln F)_x(t)\|_H^2.$$

By applying Gronwall lemma to (5.9), we see that

$$n_1(t) = n_2(t), \quad \ln f_1(t) = \ln f_2(t) \quad \text{in } V, \quad \forall t \in [0, T_0].$$

This implies that (AP) has at most one solution [n, f] on $[0, T_0]$.

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