



Jerzy Rutkowski (1920-2012)

About the Author

Prof. Jerzy Rutkowski was born on 10.03.1920 in Warsaw. He graduated from the Electrical Engineering Department of Warsaw University of Technology (WUT) in 1946 receiving MSc degree. He started academic career in 1954 having been employed at the Department of Working Machines and Vehicles (WMV) of WUT. He received PhD degree in 1960 defending the thesis "New assessment criteria of the fuel anti-detonation properties" (Promoter: prof. Jerzy Dowkontt) and habilitation degree in 1962 for the work "Chosen problems of stroboscopy theory and the design of flash stroboscopes".

The subject of his scientific interest were at the beginning the electrical and optical measurement methods tied up with the research on combustion engines at the Chair of Mechanical Vehicles Engines. The highest achievement at that time was publishing the monograph "Stroboskopy" (in Polish) by Wydawnictwa Naukowe -Techniczne in 1961, first in the world in that domain, covering the stroboscopy theory and the stroboscopes design. The book was translated later into English under the title "Stroboscopes for Industry and Research" and published by Pergamon Press in 1966. The design achievement was the elaboration of prototypes of several measurement devices: stroboscopic rotational velocity meter, the device for differentiating and marking indicator diagram, electronic device for self-ignition detection and the stand for determination of sparking plug heat factor. Prof. Rutkowski lectured "Electrical measurements of mechanical quantities" and "Electronics" at the WMV Department. He was the V-ce Dean of the WMV Department in 1964-66.

In 1962-63 he has spent a year as a Ford scholar at the Massachusetts Institute of Technology (MIT), Cambridge Mass. (7 months) and University of California (U of C), Berkeley Ca. (5 months). While at MIT he performed measurements of rapid air compression, fuel injection and combustion in Rapid Compression Machine. At U of C he investigated shock waves and detonations by the electronic and the visualization methods. He also started work on the time-space analysis of physical quantities.

During his stay in the US he led lectures and seminars: "Introduction to the Theory of Fluidics" at MIT and "The Continuity Principle for Flow Systems" at U of C. He also lectured at the research institutes and universities of Great Britain, France, Switzerland and Egypt.

In 1969 he gave a presentation "Concept of Referential Derivative in Gas Dynamics of Explosions" at the II International Colloquium "Gas Dynamics of Explosions and Reactive Systems" in Novosybirsk in the former USSR.

In 1970 he received the professor title given by the State Council. In the same year he started work at the Department of Thermodynamics of the Institute of Thermal Engineering (ITE) at the Department of Mechanical, Power and Aviation Engineering (MPAE) of WUT. In 1978-91 he was the Head of the Department of Thermodynamics and in 1983-84 the Director of the ITE.

The highest scientific achievement of prof. Rutkowski was the work in the domain of balancing mass, momentum, energy and entropy which can be named "Phenomenological modeling of physical processes". It was an attempt to integrate the physical sciences divided into mechanics, thermodynamics, electricity and magnetism. Its crowning was the book "The Basis of Balancing Mass, Momentum, Energy and Entropy" (in Polish) edited by WUT in 1976.

The English translation “The Balance Equations of Mass, Momentum, Energy and Entropy Based on the Referential Principle” was done in 1975 and has not been published yet. The electronic version of the book presented here is its first publication.

Prof. Rutkowski’s achievement was also establishing in ITE the Research Group for investigating micropolar fluid,, magnetohydrodynamics, moving interfacial barrier and the theory of phase transitions. In 1976-81 prof. Rutkowski took part in the program of the Polish-American Research Collaboration of the ITE.

He lectured “The phenomenological modeling of physical processes” at the MPAE Department.

Prof. Rutkowski’s hobby were ships and sea battles. He published several articles on the subject and among them “The Jutland Battle”. In the battle in 1916 close to Jutland Peninsula, British Grand Fleet and German Hochseeflotte were engaged. The article was published in the Illustrated Daily Courier in 1936 in Warsaw. Prof. Rutkowski cooperated with Jane’s Fighting Ships publishing data on warships of all navies around the world.

It is worth to mention the opinion ^(*) of a well known scientist prof. J. Groszkowski who is famous for decoding the control systems of V-2 rocket used by Nazi Germany during WW II:

“Jerzy Rutkowski is a type of the scientist who does not confine himself to a narrow technical specialty but continuously broadens the range of his interests in the search for new applications of solutions of formerly solved problems. His scientific achievements are characterized by the gradual transition from the experimental tools of cognition via research methods to the mining of the heart of investigated phenomena and theoretical generalizations. Such type of a scientist is needed for the Polish science today when the qualitative steps of the new technology depend on the integration of different scientific disciplines under the umbrella of technical physics”.

Prof. Jerzy Rutkowski died on 14.09.2012 in Warsaw.

(*) the opinion requested for professor title

The Balance Equations
of Mass, Momentum, Energy and Entropy
Based on the Referential Principle

By

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Robert Rudowski - nephew of Jerzy Rutkowski (1920-2012)

Preface

This book has been designed as a textbook on the phenomenological theory of balance equations for extensive quantities in non-relativistic conditions. The final goal of reasoning is to expose universal methods of formulating (though not solving) the balance equations, and to apply them to the balances of mass, momentum, energy, and entropy. Though the present work touches many problems of the continuum mechanics, thermodynamics, chemical kinetics, and electromagnetics, it does not claim to be a textbook on physics.

Some physical phenomena and laws will be presented otherwise than along traditional lines. Some formulations may seem controversial when confronted with the orthodox habits of thinking, such as the frequent use of particular and simplified models, considering stagnation and rest as a starting point for all the reasoning, identifying any velocity with the velocity of a material body, separate treatment of thermomechanical and electromagnetic phenomena, accepting historical inconsequences in a few formulations, and a tendency to generalize some old notions which preserve their univocal meaning only in particular cases.

It seems to the present author that there are three main features of his approach: a broad kinematic generalization of the balance references, the phenomenological concreteness of physical quantities and processes, and a composite treatment of thermomechanical and electromagnetic phenomena, with the traditional bulkhead of isolation completely removed.

In the classical thermodynamics we find a contrary approach. The balance references are modest, being limited to the closed and open systems in space, and to very, if not infinitely, slow processes in time. Though the thermodynamic concepts of work and heat are intended to be of widest generality, they display adequacy to

physically specific phenomena in trivial cases only, being not helped by different and different but inconsistent definitions. This is the cause why the present author remains sceptical about the "universal" formulation of the First Law of Thermodynamics (see Sec. 5.16). According to his opinion, the only definite description of work is the scalar product of force and displacement (Sec. 5.9), and the word "heat" remains univocal only when connected with a specific phenomenon (e.g. specific heat, Joule's heat, etc.). The general concepts of work and heat are not necessary at all in the balance equations which contain terms representing the substantiated conversions and transfers of energy.

In the classical electromagnetics all the physical quantities are specific at any arbitrary time-space variability of the fields, but the constitutive equations (phenomenological relations) and the balance equations are formulated under a tacit assumption of medium at rest, thus the electromechanical energy conversion cannot be revealed. Though the relativistic electrodynamics does not restrict any kinematic frames of reference, nevertheless it does not deal with any integral balances in the deformable regions.

It is supposed to be obvious in thermodynamics and fluid mechanics that the substance is the site of all extensive quantities. Such an assumption disregards the only convincing distribution model for the electromagnetic energy, which can be stored in the vacuum as well. This is why in the thermodynamics the crucial distinction between the extensive quantity and its sub-category named the substantial quantity becomes blurred. Another implication of such a way of thinking is identification of any body force with its sub-category named the mass force. But the Coulomb and Lorentz forces are of charge and current nature after all!

The nonequilibrium thermodynamics is at pains to treat all the physical phenomena (including the electromagnetic ones) on a broader basis with the help of some unspecified potential energy being connected with locally time-constant potentials and potential forces. Such a model remains true for the gravity phenomenon in terrestrial conditions and for a very particular case of stationary electromagnetics. In the time-variable fields neither the electrodynamic force nor the electromagnetic energy preserve their potential character.

There is no need, however, to use any anonymous physical quantities in the specific balance equations, provided we had a deeper look into the essence of phenomenological model of matter discussed in Chapter 2 of the present book. If the so-called apparent limit (denoted by a new symbol $\lim f$) does exist, then we are allowed to classify the motions of substance and the force fields distinctly. It leads next to the specification of 4 and only 4 phenomenologically defined kinds of energy.

Chapter 3 contains a brief sketch of the theory of kinematic references for the balances. It deals with the behaviour of physical fields in arbitrarily moving geometric objects called the referential objects (as they were named by the present author in 1962, see Bibliography 1, 2). In physical applications, an abstractive referential region may take specific shape of the substantial or componential region, the combustion zone, etc., each of them having definite geometric and physical properties.

Consequently, the well-known substantial derivative (frequently used in the fluid mechanics) is a particular case of the referential derivative, and the theorems of Reynolds (concerning the transport of substantial quantity), Helmholtz (on the fluid vorticity), and Thomson (on the fluid velocity circulation) are nothing else than applications of the Leibniz's referential trans-

formation group. At this stage, the so-called time-space operators show their utility. One of them, the Helmholtzian, is already known in the fluid mechanics. The remaining two have been named the Reynoldsonian and the Thomsonian by the present author.

The concept of Reynoldsonian is of fundamental importance to any differential balance equation. It is not the time-derivative but the Reynoldsonian that describes the rate of change of an extensive quantity at a moving point.

The scope of application of the referential theory outgrows the problems of balancing. It covers the general field theory, as well as all the branches of phenomenological physics.

The referential Strouhal number shows its usefulness as a criterion for checking the fields whether they can be considered as quasi-stationary or quasi-homogenous (Sec. 3.9). The general laws ruling the switch-over of kinematic references allow very simple deduction of equations for the relative flow, and for the motion of rocket (see Exs. 5.13, 14 in Sec. 5.8). Lastly, in Sec. 3.10 (Ex. 3.20), the author has explained how the relation between the electric field intensity and the velocity of reference point can be obtained by means of the Leibniz-Reynolds referential transformation, without use of the relativistic Lorentz transformation.

Chapter 4 begins with the discussion of the basic balance concepts of storage, production, and transfer of an extensive quantity. It has been emphasized that the rate of change of storage, as a derivative, and the transfer, as a process subjected to the capture effect (Eq. 4.20), are of referential character.

When discussing the concept of the transfer-flux density, hitherto explained by particular examples of the mass and momentum transport, the author used the model of directional transmission, being adopted from photometry. Owing to it, the simple algebraic transformation (4.13b) makes it easier to understand why the

transfer-flux density of a vectorial extensive quantity must be a second-rank tensor.

On account of unification of the transfer terms in all the specific balances and phenomenological relations, the present author has followed Bird and others in using the stress sign convention opposed to the traditional one (Sec. 4.8). This apparently shocking replacement (as well as using the viscosity coefficient twice as large as the conventional one, Sec. 5.6) is but a correction of historical inconsequence.

So far, too little attention has been drawn to the notions of storage, production, and transfer. The ignorance of mentioned terms brings about, in traditional textbooks of thermodynamics, half-page verbal descriptions of the Second Law of Thermodynamics, though it is possible to replace them by a one-line universal mathematical inequality (Eq. 7.23). The indistinction between storage and production in the chemical kinetics leads to erroneous (in the general case) presentation of the reaction rate as a total derivative of concentration (Sec. 6.1), and to wordy descriptions of the differences between the kinetic and diffusive combustions, while they can be replaced with two short formulae (6.20). It is due to the ignorance of the Leibniz referential transformations that in some works we find the balance equations of component's momentum with false physical interpretation (Sec. 6.8).

As mentioned above, it is very common to place the equality sign between the extensive and substantial quantity. According to the present author's classification, the substantial and componential quantities are but particular cases of an extensive quantity. For each of those quantities the author has set together the definitions of various categories of densities, transformations of the Reynoldsians, and classifications of different kinds of transfer (e.g. the convective, non-convective, diffusive, non-diffusive).

The substantial character of reference in all the phenomenological relations (which describe purely non-convective transfer) has been emphasized. For example the traditional denotation of Ohm's law is, generally, not true, keeping its validity only in the particular case of resting substance (Sec. 4.14).

The present author has endeavoured to emphasize the practical importance of the theorem on equivalence of balances (Sec. 4.10). Universal recipes allow direct change from one form of a balance equation to another, according to the switch-over of the kinematic reference, for any physically specified extensive quantity, with no need of arduous mathematical transformations in each individual case. It would be impossible to work out the recipes mentioned without using the concepts of the referential velocity, referential derivative, and the Reynoldsian.

This kind of approach to the balance theory has been elaborated during 1962-1972, and is presented herein for the first time as a systematic whole. The theory is consistent to the effect that there are various ways leading to the same mathematical formulations. For example it is possible to deduce Eqs.(4.20) for the capture effect by means of geometric and kinematic reasoning, as well as with the help of a purely analytical method (Ex. 4.3 in Sec. 4.10).

In the balances of momentum, energy, and entropy, the present author takes the electromagnetic phenomena into full account, without imposing any restrictions save for the assumption of the non-relativistic conditions. The electromagnetic terms in the balances mentioned are fully specified. They originate from Maxwell's equations and the so-called improved balance of electromagnetic energy (Sec. 4.15), deduced by the present author in 1970. This equation, valid for any kind of medium, reveals the effects of the hysteresis and electromechanical conversion. It also discerns the energy transfer by means of conduction of current and radiation of waves.

Since all the phenomenologically crucial effects are specified, it is possible to make a synthetic review of various kinds of the energy conversion and transfer (Sec. 7.2), and to examine the energy balance of a universal electro-roto-flow system (Sec. 7.5). Such a discussion seems purposeful in view of the prospective development of the new sources of energy and propulsion.

The applications of the general balance theory contained in this book are not complete. There was no space available to consider the balances of the electromagnetic momentum and the intrinsic angular momentum. For the same reason, the intrinsic rotation in the polar fluid could not be taken into account in the balances of momentum, kinetic energy, and electromagnetic energy. Chapter 6 on the balances of componential quantities is abridged. The present author's balance equation (6.56) for the componential kinetic energy leads to the balance of the diffusion energy, which has deliberately been deleted for economy of space. The theory of balances for the discontinuity surfaces (e.g. for the detonation wavefront) has been omitted completely.

Despite the customary way, the chapter on the mathematical symbols, definitions and relations has been placed at the beginning of the book. This is because all the exposition is written by means of an unconventional bar system notation for the vectors and tensors, supplemented with the arrow symbols of referentiality. Its simple rules should be acquired by the reader before studying the virtual contents of the book.

The choice of notation resulted from long considerations. In a work dealing with numerous quantities of every branch of physics, nearly all the resources of the Roman and Greek alphabets are consumed for the kernel and suffix letters. In such circumstances the use of the customary index notation for the vectors and tensors would prove troublesome.

On the other hand, the print (Gibbs') notation, with boldface letters, is very strenuous as applied to the "chalk and blackboard" practice, handwriting, and adjusting a typescript.

In the mentioned system, the vectors and tensors are hardly distinguishable each from the other, and the denotation of transposes and transpositional operators is not as simple as in the bar system. None of the index and print notations reveals the tensorial rank of a product, or differential operator, at first sight.

There was also another problem to solve. Since the balances may have references to differently moving points and regions, it was required to introduce special symbols for the referentiality, total derivatives of various types, and the time-space operators.

The use of different letters or suffices for the referential and non-referential quantities would be wasting the nearly exhausted contents of both alphabets. It would be also an irrational fortuity to replace d with D , or to put dots or accent signs, in order to make the substantial and componential derivatives distinguishable.

For all those reasons an adaptation of either of the most popular notations would create an inconsequent and confused hybrid system. To avoid this, the present author has elaborated (1959-66) the bar system notation, principally based on a standard typewriter keyboard, and perfectly fit to the "chalk and blackboard" use.

In his own system, the author makes use of the prefixes as symbols for all the differential operators. In the denotations of old operators, such as the divergence, the prefixes are the traditional abbreviations of Latin words (for that reason the author favours rot instead of the familiar but non-Latin curl). The new time-space operators, such as the Reynoldsian, are denoted with prefixes originating from scientists' names. The author does not favour the

nabla symbol, since it is non-applicable to the transpositional operators (see Sec. 1.5).

The principle of the proper number of the integral signs, as well as of the proper infinitesimality order (Sec. 1.4), has been strictly observed. This is not a pedantry but a necessity, especially in Secs. 3.12, 13, where we are dealing with infinitesimally thin slices and infinitesimally narrow stripes. And, as far as the volume is concerned, the thermodynamic dV is something different from d^3V appearing in many integrals (Sec. 5.14).

The present textbook has resulted from 4 years of lectures of the subject "Principles of Balance Equations" at the Warsaw Technical University. Practical teaching induced the author to save some space in the book by transferring several proofs, deductions, transformations, and applications to 70 exercises. Save for the easiest ones, they are supplied with the indispensable instructions and results.

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Note: The rough-designs of 26 figures: 2.1 to 2.3, 3.1 to 3.7, 4.1 to 4.6, 5.1 to 5.5, 6.1 to 6.3, 7.1 and 7.2 see pages F 1 to F 9.

1. Basic Mathematical Symbols, Definitions and Relations

1.1. Notation

The bar system notation for vectors and tensors has been used in the present exposition for the easiness of implementation in a standard typescript, as well as in the practice of "chalk and blackboard".

The tensor rank corresponds to the number of bars over a letter symbol (a scalar - no bar, a vector - single bar, a tensor - double bar). In the notation of a product all the factors are covered with as many bars as the tensorial rank of the multiplication result is. Only the scalar product is deprived of the common bar but enclosed within parentheses (which may be also used to enclose the terms being added or multiplied in an ordinary way, provided it does not bring confusion). The cross \times between the factors indicates the alternating anticommutability of the product, the dot between the upper bars of the factors marks the transpositional anticommutability. The denotation of a differential operator (gradient, divergence, curl, Reynoldsonian, Helmholtzian, Thomsonian) contains as many bars over the prefix as the tensorial rank of the result of operation is.

At more important definitions the symbols in the usual print (Gibbs') notation and the index notation will be given for comparative purposes.

The print notation (abbreviation: [print]) distinguishes tensors of various ranks by different characters (regular for a scalar, boldface for a vector, narrow boldface for a tensor). The sign between the factors indicates the type of product and the reduction of tensorial rank of the multiplication result with respect to the sum of the factor ranks: the cross \times (reduction by 1), the dot \cdot (by 2), and the double dot \colon (by 4). The lack of any sign

means preservation of tensorial rank. The differential operators are denoted by the association of symbolic vector nabla ∇ (with Cartesian components $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$) with an appropriate multiplication sign.

In the index notation (abbreviation: $[\text{index}]$) the tensor of any rank (including the product and differential operator) is written in form of its symbolic component, distinguished by a group of the coordinate indices (i, j, k, \dots) at the kernel letter or letters. The non-repeated (free) coordinate indices identify the symbolic component, the number of them being equal to the tensor rank. The indices repeated in pairs (dummy indices) symbolize the shortened notation of a sum according to Einstein's convention. In the notation of products and differential operators some auxiliary operators are used: the Kronecker delta δ_{ij} (equal to 0 when 2 among 3 indices are identical, and to 1 for 3 different indices), and the alternating triadic ϵ_{ijk} (equal to 0 when 2 among 3 indices are identical, and to ± 1 for 3 different indices, at their sequence the same or opposite as with respect to the cyclic order of coordinates, respectively). The differential operators may be denoted with the comma system (after the last index of the differentiated expression we put a comma and the index of the differentiation variable).

At the definitions of vectors, tensors, their products and differential operators, the matrices of their specific components will be given by way of example of the Cartesian coordinate system x, y, z , in the ordinary three-dimensional space.

The prefix Δ indicates an interval or increment, and δ a small interval or increment. In order to emphasize that some quantity α is a function of quantity β , the latter will be enclosed with the angle brackets: $\alpha\langle\beta\rangle$.

1.2. Scalars, Vectors, and Tensors

(1.1) A scalar (zeroth rank tensor) has $3^0 = 1$ component α

$$(1.2) \left\{ \begin{array}{l} \text{A vector (first} \\ \text{rank tensor) has} \\ 3^1 = 3 \text{ components} \end{array} \right\} \quad \bar{A} = \left\| \begin{array}{c} A_x \\ A_y \\ A_z \end{array} \right\|, \quad \mathbf{A} \text{ [print]}, \quad A_i \text{ [index]}$$

Any vector \bar{A} is the product of its scalar magnitude A and its unit vector \bar{I}_A (having the dimensionless magnitude of 1):

$$(1.3) \quad \bar{A} = \bar{I}_A A$$

A self-contained unit vector points some direction (with the sense included) in space.

$$(1.4) \left\{ \begin{array}{l} \text{A tensor (of the} \\ \text{second rank) has} \\ 3^2 = 9 \text{ components} \end{array} \right\} \quad \bar{a} = \left\| \begin{array}{ccc} a_{xx}, a_{xy}, a_{xz} \\ a_{yx}, a_{yy}, a_{yz} \\ a_{zx}, a_{zy}, a_{zz} \end{array} \right\|, \quad a \text{ [print]}, \quad a_{ij} \text{ [index]}$$

$$(1.5a) \left\{ \begin{array}{l} \text{The unit} \\ \text{tensor} \end{array} \right\} \quad \bar{I} = \left\| \begin{array}{ccc} 1, 0, 0 \\ 0, 1, 0 \\ 0, 0, 1 \end{array} \right\|, \quad \mathbf{I} \text{ [print]}, \quad \delta_{ij} \text{ [index]} \quad \left\{ \begin{array}{l} \text{(the} \\ \text{Kronecker} \\ \text{Delta)} \end{array} \right.$$

$$(1.5b) \left\{ \begin{array}{l} \text{The spherical} \\ \text{tensor} \end{array} \right\} \quad \bar{I}\alpha = \left\| \begin{array}{ccc} \alpha, 0, 0 \\ 0, \alpha, 0 \\ 0, 0, \alpha \end{array} \right\|$$

(1.6) If the tensors \bar{a} and \bar{b} have the components $a_{ij} = b_{ji}$ then one of the tensors is the transpose (symbol: \neq) of the other, and vice versa: $\bar{a} = \bar{b}'$, $\bar{b} = \bar{a}'$.

(1.7a) If the components $a_{ij} = +a_{ji}$ then the tensor \bar{a} is symmetric.

(1.7b) If the components $a_{ij} = -a_{ji}$ then the tensor \bar{a} is anti-symmetric.

(1.8a) The symmetric part \bar{a}^s of tensor \bar{a} :

$$\bar{a}^s = \frac{1}{2}(\bar{a} + \bar{a}'), \quad \frac{1}{2}(a_{ij} + a_{ji}) \text{ [index]}$$

(1.8b) The antisymmetric part \bar{a}^a of tensor \bar{a} :

$$\bar{a}^a = \frac{1}{2}(\bar{a} - \bar{a}'), \quad \frac{1}{2}(a_{ij} - a_{ji}) \text{ [index]}$$

(1.9) The decomposition of a tensor into symmetric and antisymmetric part:

$$\bar{a} = \bar{a}^s + \bar{a}^a, \quad a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji}) \quad [\text{index}]$$

$$\bar{a}^a = \bar{a}^s - \bar{a}^a$$

(1.10a) The symmetry symbol for a tensor: $\bar{a} = \bar{a}^s$ (since $\bar{a}^a = 0$).

(1.10b) The antisymmetry symbol for a tensor: $\bar{a} = \bar{a}^a$ (since $\bar{a}^s = 0$).

(1.11a) The scalar or the trace of a tensor:

$$\text{Tr } \bar{a} = a_{xx} + a_{yy} + a_{zz}, \quad a_{ii} \quad [\text{index}]$$

$$\text{Tr } \bar{a} = \text{Tr } \bar{a}^a = \text{Tr } \bar{a}^s, \quad \text{Tr } \bar{a}^a = 0$$

(1.11b) $\text{Tr } \bar{I} = 3, \quad \text{Tr}(\bar{I}\alpha) = 3\alpha$

(1.12) The (pseudo-) vector of a tensor \bar{a} = $\left\| \begin{array}{l} \frac{1}{2}(a_{zy} - a_{yz}) \\ \frac{1}{2}(a_{xz} - a_{zx}) \\ \frac{1}{2}(a_{yx} - a_{xy}) \end{array} \right\|, \quad \frac{1}{2}(a_{ji} - a_{ij}) \quad [\text{index}]$

Note: in a more frequent use is another definition, viz.

$\frac{1}{2}(a_{ij} - a_{ji}) \quad [\text{index}]$, corresponding to $-\text{Vec } \bar{a}$ in the bar system notation.

(1.13a) The deviator (or deviatorial part) \bar{a}^d of a tensor \bar{a} :

$$\bar{a}^d = \bar{a} - \frac{1}{3} \text{Tr } \bar{a}, \quad \bar{a}^d a = \bar{a}^a, \quad \text{Tr } \bar{a}^d = 0$$

(1.13b) The decomposition of a tensor into spherical, deviatorial-symmetric and antisymmetric part: $\bar{a} = \frac{\bar{I}}{3} \text{Tr } \bar{a} + \bar{a}^d s + \bar{a}^a$

1.3. The Products of Vectors and Tensors

(1.14a) The scalar product of vectors \bar{A} and \bar{B} (a scalar):

$$(\bar{A} \bar{B}) = (\bar{B} \bar{A}) = A_x B_x + A_y B_y + A_z B_z, \quad \bar{A} \cdot \bar{B} \quad [\text{print}], \quad A_i B_i \quad [\text{index}]$$

(1.14b) The square of vector \bar{A} (a scalar): $(\bar{A} \bar{A}) = (\bar{A}^2) = A^2$

(1.15a) The vector product of vector \bar{A} by vector \bar{B} (a vector):

$$\overline{\bar{A} \times \bar{B}} = - \overline{\bar{B} \times \bar{A}} = \left\| \begin{array}{l} A_y B_z - A_z B_y \\ A_z B_x - A_x B_z \\ A_x B_y - A_y B_x \end{array} \right\|, \quad \bar{A} \times \bar{B} \quad [\text{print}], \quad \epsilon_{ijk} A_i B_j \quad [\text{index}]$$

(1.15b) $\overline{\bar{A} \times \bar{A}} = 0$

(1.16a) The dyadic product of vector \bar{A} by vector \bar{B} (a tensor):

$$\bar{\bar{A}} \cdot \bar{\bar{B}} = \bar{\bar{B}} \cdot \bar{\bar{A}} = \begin{pmatrix} A_x B_x & A_x B_y & A_x B_z \\ A_y B_x & A_y B_y & A_y B_z \\ A_z B_x & A_z B_y & A_z B_z \end{pmatrix}, \quad \underline{AB} \text{ [print]}, \quad A_i B_j \text{ [index]}$$

$$\text{Tr } \bar{\bar{A}} \cdot \bar{\bar{B}} = (\bar{A} \cdot \bar{B}), \quad \overline{\text{Vec}} \bar{\bar{A}} \cdot \bar{\bar{B}} = -\frac{1}{2} \overline{\bar{A} \times \bar{B}}$$

(1.16b) The dyadic of vector \bar{A} (a tensor): $\bar{\bar{A}} \cdot \bar{\bar{A}}$

$$\text{Tr } \bar{\bar{A}} \cdot \bar{\bar{A}} = (\bar{A}^2) = \bar{A}^2, \quad \overline{\text{Vec}} \bar{\bar{A}} \cdot \bar{\bar{A}} = 0$$

(1.17a) The scalar product of tensors $\bar{\bar{a}}$ and $\bar{\bar{b}}$ (a scalar):

$$(\bar{\bar{a}} \cdot \bar{\bar{b}}) = (\bar{\bar{b}} \cdot \bar{\bar{a}}) =$$

$$a_{xx}b_{xx} + a_{xy}b_{xy} + a_{xz}b_{xz} + a_{yx}b_{yx} + a_{yy}b_{yy} + a_{yz}b_{yz} + a_{zx}b_{zx} + a_{zy}b_{zy} + a_{zz}b_{zz}, \\ a_{ij}b_{ij} \text{ [index]}$$

Note: in a more frequent use is another definition, viz.

$a_{ij}b_{ji}$ [index], corresponding to $\underline{a:b}$ [print] and $(\bar{\bar{a}} \cdot \bar{\bar{b}})$ in the bar system notation; if $\bar{\bar{b}} = \bar{\bar{b}}^s$, and consequently $\bar{\bar{b}} = \bar{\bar{b}}^t$, then $(\bar{\bar{a}} \cdot \bar{\bar{b}}) = (\bar{\bar{a}} \cdot \bar{\bar{b}}^t)$.

(1.17b) The square of tensor $\bar{\bar{a}}$ (a scalar): $(\bar{\bar{a}} \cdot \bar{\bar{a}}) = (\bar{\bar{a}}^2)$

(1.17c) $(\bar{\bar{I}} \cdot \bar{\bar{a}}) = \text{Tr } \bar{\bar{a}}$, (1.17d) $(\bar{\bar{I}} \cdot \bar{\bar{I}}) = (\bar{\bar{I}}^2) = \text{Tr } \bar{\bar{I}} = 3$

(1.17e) $(\bar{\bar{a}} \cdot \bar{\bar{b}}) = (\bar{\bar{a}}^s \cdot \bar{\bar{b}}^s)$, provided $\bar{\bar{a}} = \bar{\bar{a}}^s$, since $(\bar{\bar{a}}^s \cdot \bar{\bar{b}}^a) = 0$.

(1.17f) $(\bar{\bar{a}}^{ds} \cdot \bar{\bar{a}}) = (\bar{\bar{a}}^{ds2})$, since $(\bar{\bar{I}} \cdot \bar{\bar{a}}^{ds}) = \text{Tr } \bar{\bar{a}}^{ds} = 0$, $(\bar{\bar{a}}^{ds} \cdot \bar{\bar{a}}^a) = 0$.

(1.18a) The affinor product of vector \bar{A} by tensor $\bar{\bar{a}}$ (a vector):

$$\overline{\bar{A} \cdot \bar{\bar{a}}} = \overline{\bar{\bar{a}} \cdot \bar{A}} = \begin{pmatrix} A_x a_{xx} + A_y a_{yx} + A_z a_{zx} \\ A_x a_{xy} + A_y a_{yy} + A_z a_{zy} \\ A_x a_{xz} + A_y a_{yz} + A_z a_{zz} \end{pmatrix}, \quad \underline{A \cdot a} \text{ [print]}, \quad A_i a_{ij} \text{ [index]}$$

(1.18b) The affinor product of tensor $\bar{\bar{a}}$ by vector \bar{A} (a vector):

$$\overline{\bar{\bar{a}} \cdot \bar{A}} = \overline{\bar{A} \cdot \bar{\bar{a}}} = \begin{pmatrix} a_{xx}A_x + a_{xy}A_y + a_{xz}A_z \\ a_{yx}A_x + a_{yy}A_y + a_{yz}A_z \\ a_{zx}A_x + a_{zy}A_y + a_{zz}A_z \end{pmatrix}, \quad \underline{a \cdot A} \text{ [print]}, \quad a_{ij}A_j \text{ [index]}$$

The vectors \bar{A} , $\overline{\bar{A} \cdot \bar{\bar{a}}}$, $\overline{\bar{\bar{a}} \cdot \bar{A}}$ have, in general, different directions (see Fig. 4.6 in Sec. 4.7).

$$(1.18c) \quad \overline{\overline{A}^* \overline{a}} = \overline{\overline{a}^* \overline{A}}, \text{ provided } \overline{a} = \overline{a}^S, \overline{a} = \overline{a}^{\prime}$$

$$(1.18d) \text{ The identity product of a vector: } \overline{\overline{A}^* \overline{I}} = \overline{\overline{I}^* \overline{A}} = \overline{A}$$

$$(1.19a) \text{ The mixed product of 3 vectors: } (\overline{A} \overline{B} \times \overline{C}) = (\overline{A} \times \overline{B} \overline{C})$$

$$(1.19b) \quad (\overline{A} \overline{A} \times \overline{B}) = 0$$

$$(1.20a) \quad (\overline{A} \overline{B}) \overline{C} = \overline{\overline{A} \overline{B}^* \overline{C}}, \quad (1.20b) \quad \overline{A} (\overline{B} \overline{C}) = \overline{\overline{A}^* \overline{B} \overline{C}}$$

$$(1.21) \quad (\overline{\overline{A}^* \overline{C}} \overline{B}) = (\overline{A} \overline{\overline{C}^* \overline{B}}) = (\overline{\overline{A}^* \overline{B} \overline{C}})$$

1.4. The Geometric and Field Quantities

$$(1.22) \left\{ \begin{array}{l} \text{The position vector} \\ \text{of a point in ordinary} \\ \text{3-dimensional space} \end{array} \right\} \quad \overline{r} = \left\| \begin{array}{c} x \\ y \\ z \end{array} \right\|$$

(1.23a) A line l consists of infinitesimal elements $d\overline{l}$.

(1.23b) A surface s consists of infinitesimal elements $d^2\overline{s} = \overline{I}_s d^2s$,
where \overline{I}_s is the unit vector (normal to the surface), and
 d^2s is the magnitude (scalar area).

(1.23c) The nonparallel elements $d\overline{l}_1$ and $d\overline{l}_2$ form a surface
element $d^2\overline{s} = \overline{d\overline{l}_1 \times d\overline{l}_2}$.

(1.23d) A spatial region V consists of infinitesimal cells of vol-
ume $d^3V = (d\overline{l} d^2\overline{s})$, formed by the nonparallel vector ele-
ments $d\overline{l}$ and $d^2\overline{s}$.

An upper index at d indicates the infinitesimality order of
geometric element, i.e. the number of its infinitesimal dimension-
ality. For the sake of convenience the index 1 will be omitted,
e.g. we write $d\overline{l}$ instead of $d^1\overline{l}$. The area of an infinitesimally
narrow stripe (see Fig. 3.7 in Sec. 3.13) is $d^1s = ds$. The volume
of an infinitesimally thin slice (see Fig. 3.6 in Sec. 3.12) is
 $d^1V = dV$.

The integral over a geometric object gives non-infinitesimal
value, provided the multiplicity of integration (of the sign \int)
is equal to the infinitesimality order of geometric element:

$$(1.24a) \left. \begin{array}{l} \text{Integral along} \\ \text{open line} \end{array} \right\} \int_1 \dots d\Gamma, \quad (1.24b) \left. \begin{array}{l} \text{Integral along} \\ \text{closed line,} \\ \text{or contour} \end{array} \right\} \oint_1 \dots d\Gamma$$

$$(1.24c) \left. \begin{array}{l} \text{Integral} \\ \text{over open} \\ \text{surface} \end{array} \right\} \iint_s \dots d^2\bar{s}, \quad (1.24d) \left. \begin{array}{l} \text{Integral} \\ \text{over closed} \\ \text{surface,} \\ \text{or shell} \end{array} \right\} \oint_s \dots d^2\bar{s}$$

$$(1.24e) \text{ Integral over spatial region } \iiint_V \dots d^3V$$

The expressions for specific components of vectors and tensors are based on the principles of:

(1) right-handed orientation between any open surface s and its constour l ,

(2) outward orientation of any shell s with respect to enclosed spatial region V .

A field quantity (e.g. the temperature, or the fluid velocity) is function of its location defined by the position vector \bar{r} , and time t . It has a spatial distribution and a time process, so it forms a time-space field. We shall assume this field to be continuous and differentiable always and everywhere, unless an appropriate provision is made.

A non-field quantity can be located in a single selected point or region (treated as an entirety), mobile or immobile. It has no spatial distribution and is a function of time only (e.g. the velocity of an isolated material point, the relative position of two selected points of a deforming elastic body).

To the non-field quantities belong the integral quantities, i.e. the integrals of field quantities over mobile or immobile geometric objects. As a result of spatial integration, the described type of quantity remains a function of time only (e.g. the momentum of a definite body).

We assume that the spatial differentiation and integration of the field quantities is taking place at a "frozen" time instant $t = \text{const.}$

1.5. Differential Operators of the Space Fields

$$(1.25a) \left. \begin{array}{l} \text{The gradient} \\ \text{of scalar } \alpha \\ \text{(a vector)} \end{array} \right\} \overline{\text{grad}} \alpha = \lim_{V \rightarrow 0} \frac{1}{V} \oint_S d^2 \overline{s} \alpha = \left\| \begin{array}{c} \frac{\partial \alpha}{\partial x} \\ \frac{\partial \alpha}{\partial y} \\ \frac{\partial \alpha}{\partial z} \end{array} \right\|$$

$$\nabla \alpha \text{ [print]}, \quad \alpha_{,i} \text{ [index]}$$

$$(1.25b) \left. \begin{array}{l} \text{The gradient} \\ \text{of vector } \overline{A} \\ \text{(a tensor)} \end{array} \right\} \overline{\text{grad}} \overline{A} = \lim_{V \rightarrow 0} \frac{1}{V} \oint_S d^2 \overline{s} \overline{A} = \left\| \begin{array}{ccc} \frac{\partial A_x}{\partial x}, & \frac{\partial A_y}{\partial x}, & \frac{\partial A_z}{\partial x} \\ \frac{\partial A_x}{\partial y}, & \frac{\partial A_y}{\partial y}, & \frac{\partial A_z}{\partial y} \\ \frac{\partial A_x}{\partial z}, & \frac{\partial A_y}{\partial z}, & \frac{\partial A_z}{\partial z} \end{array} \right\|$$

$$\nabla A \text{ [print]}, \quad A_{i,j} \text{ [index]}$$

$$(1.25c) \left. \begin{array}{l} \text{The divergence} \\ \text{of vector } \overline{A} \\ \text{(a scalar)} \end{array} \right\} \text{div } \overline{A} = \lim_{V \rightarrow 0} \frac{1}{V} \oint_S (d^2 \overline{s} \overline{A}) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \cdot A \text{ [print]}, \quad A_{i,i} \text{ [index]}$$

$$(1.25d) \left. \begin{array}{l} \text{The divergence} \\ \text{of tensor } \overline{a} \\ \text{(a vector)} \end{array} \right\} \overline{\text{div}} \overline{a} = \lim_{V \rightarrow 0} \frac{1}{V} \oint_S d^2 \overline{s} \overline{a} = \left\| \begin{array}{ccc} \frac{\partial a_{xx}}{\partial x} + \frac{\partial a_{yx}}{\partial y} + \frac{\partial a_{zx}}{\partial z} \\ \frac{\partial a_{xy}}{\partial x} + \frac{\partial a_{yy}}{\partial y} + \frac{\partial a_{zy}}{\partial z} \\ \frac{\partial a_{xz}}{\partial x} + \frac{\partial a_{yz}}{\partial y} + \frac{\partial a_{zz}}{\partial z} \end{array} \right\|$$

$$\nabla \cdot a \text{ [print]}, \quad a_{ij,i} \text{ [index]}$$

$$(1.25e) \left. \begin{array}{l} \text{The curl,} \\ \text{or rotation,} \\ \text{of vector } \overline{A} \\ \text{(a vector)} \end{array} \right\} \overline{\text{rot}} \overline{A} = \lim_{V \rightarrow 0} \frac{1}{V} \oint_S d^2 \overline{s} \times \overline{A} = \left\| \begin{array}{cc} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{array} \right\|$$

$$\nabla \times A \text{ [print]}, \quad \epsilon_{ijk} A_{j,i} \text{ [index]}$$

$$(1.25f) \text{ The symbol of curl (or rotation) of tensor } \overline{a} \text{ (a tensor):}$$

$$\overline{\text{rot}} \overline{a}$$

(its definitional formula contains an integrated alternating-type tensor product of vector $d^2 \overline{s}$ by tensor \overline{a})

The transpositional operators:

$$(1.26a) \text{ The antigradient of vector: } \overline{\overline{\text{grad}}} \overline{A} = \overline{\text{grad}} \overline{A}$$

$$(1.26b) \text{ The antidivergence of tensor: } \overline{\overline{\text{div}}} \overline{a} = \overline{\text{div}} \overline{a}$$

$$(1.26c) \text{ The anticurl of tensor: } \overline{\overline{\text{rot}}} \overline{a} = - \overline{\text{rot}} \overline{a}$$

$$\begin{aligned}
 (1.27) \quad & \left. \begin{array}{l} \text{The affinor} \\ \text{gradient} \\ \text{(a vector)} \end{array} \right\} \quad \overline{B \cdot \overline{\text{grad}} A} = \left(\begin{array}{l} B_x \frac{\partial A_x}{\partial x} + B_y \frac{\partial A_x}{\partial y} + B_z \frac{\partial A_x}{\partial z} \\ B_x \frac{\partial A_y}{\partial x} + B_y \frac{\partial A_y}{\partial y} + B_z \frac{\partial A_y}{\partial z} \\ B_x \frac{\partial A_z}{\partial x} + B_y \frac{\partial A_z}{\partial y} + B_z \frac{\partial A_z}{\partial z} \end{array} \right) \\
 & B \cdot \nabla A \text{ [print]}, B_i A_{j,i} \text{ [index]}
 \end{aligned}$$

Transformations of differential operators:

$$(1.28a) \quad \overline{\text{grad}(\overline{A} \cdot \overline{B})} = \overline{A \cdot \overline{\text{grad}} \overline{B}} + \overline{B \cdot \overline{\text{grad}} \overline{A}}$$

$$(1.28b) \quad \overline{\text{grad}(\overline{A}^2)} = 2 \overline{A \cdot \overline{\text{grad}} \overline{A}}, \quad (1.28c) \quad \overline{\text{grad} \frac{1}{\alpha}} = -\frac{1}{\alpha^2} \overline{\text{grad} \alpha}$$

$$(1.29a) \quad \text{Tr} \overline{\text{grad}} \overline{A} = (\overline{I} \overline{\text{grad}} \overline{A}) = \text{div} \overline{A}$$

$$(1.29b) \quad \overline{\text{Vec}} \overline{\text{grad}} \overline{A} = -\frac{1}{2} \overline{\text{rot}} \overline{A}$$

$$(1.30a) \quad \overline{\text{div}(\alpha \overline{A})} = \alpha \overline{\text{div}} \overline{A} + (\overline{A} \overline{\text{grad}} \alpha)$$

$$(1.30b) \quad \overline{\text{div}} \overline{a} \cdot \overline{A} = (\overline{A} \overline{\text{div}} \overline{a}) + (\overline{a} \overline{\text{grad}} \overline{A})$$

$$(1.30c) \quad \overline{\text{div}} \overline{A} \times \overline{B} = (\overline{B} \overline{\text{rot}} \overline{A}) - (\overline{A} \overline{\text{rot}} \overline{B}), \quad (1.30d) \quad \overline{\text{div}} \overline{\text{rot}} \overline{A} = 0$$

$$(1.31a) \quad \overline{\text{div}}(\alpha \overline{I}) = \overline{\text{grad}} \alpha, \quad (1.31b) \quad \overline{\text{div}} \overline{A} \cdot \overline{B} = \overline{B} \overline{\text{div}} \overline{A} + \overline{A \cdot \overline{\text{grad}} \overline{B}}$$

$$(1.31c) \quad \overline{\text{div}} \overline{\text{grad}}^s \overline{A} = \frac{1}{2}(\overline{\text{div}} \overline{\text{grad}} \overline{A} + \overline{\text{grad}} \overline{\text{div}} \overline{A})$$

$$(1.31d) \quad \overline{\text{div}} \overline{\text{rot}} \overline{a} = 0$$

$$(1.32a) \quad \overline{\text{rot}} \overline{A} \times \overline{B} = \overline{B \cdot \overline{\text{grad}} \overline{A}} - \overline{A \cdot \overline{\text{grad}} \overline{B}} + \overline{A} \overline{\text{div}} \overline{B} - \overline{B} \overline{\text{div}} \overline{A}$$

$$(1.32b) \quad \overline{\text{rot}} \overline{\text{grad}} \alpha = 0, \quad (1.32c) \quad \overline{\text{rot}} \overline{\text{rot}} \overline{A} = \overline{\text{grad}} \overline{\text{div}} \overline{A} - \overline{\text{div}} \overline{\text{grad}} \overline{A}$$

$$(1.32d) \quad \overline{\text{rot}} \overline{\text{div}} \overline{\text{grad}} \overline{A} = \overline{\text{div}} \overline{\text{grad}} \overline{\text{rot}} \overline{A}$$

$$(1.33) \quad \overline{B \cdot (\overline{\text{grad}} \overline{A} - \overline{\text{grad}} \overline{A})} = \overline{B \times \overline{\text{rot}} \overline{A}}$$

1.6. Integral Transformations of the Space Fields

The Gauss-Green-Ostrogradsky theorem (abbreviation: GGO):

$$(1.34a) \quad \oint_S (d^2 \overline{s} \cdot \overline{A}) = \iiint_V \overline{\text{div}} \overline{A} d^3 V, \quad (1.34b) \quad \oint_S d^2 \overline{s} \cdot \overline{a} = \iiint_V \overline{\text{div}} \overline{a} d^3 V$$

The Stokes theorem:

$$(1.35) \quad \oint_l (d\overline{l} \cdot \overline{A}) = \iint_S (d^2 \overline{s} \cdot \overline{\text{rot}} \overline{A})$$

The integral along an open line beginning and ending at points 1 and 2, respectively (at $t = \text{const}$, so $d\bar{l}$ being not the displacement!):

$$(1.36a) \int_1^2 (d\bar{l} \overline{\text{grad } \alpha}) = \alpha_2 - \alpha_1, \quad (1.36b) \int_1^2 \overline{d\bar{l} \cdot \overline{\text{grad } A}} = \bar{A}_2 - \bar{A}_1$$

Small field increments $\delta\alpha$ and $\delta\bar{A}$ along a small line segment $\delta\bar{l}$:

$$(1.37a) \delta\alpha = (\delta\bar{l} \overline{\text{grad } \alpha}), \quad (1.37b) \delta\bar{A} = \overline{\delta\bar{l} \cdot \overline{\text{grad } A}}$$

2. Phenomenological Model of Structure and Processes of Matter

2.1. Phenomenological Model Versus Microphysical Reality

The kind of mathematical description of physical phenomena depends on the assumed model of structure of matter. In reality, the microstructure of substance is of grain nature (molecules, atoms, elementary particles), and the physical processes are of quantum character (collisions, electron transitions, etc.). As far as the technology is concerned, however, only the phenomenological results of physical processes play any role. They reveal in a macroscopic scale which blurs all the individual properties of the grains, as well as the very grain-structure of matter.

Within the time-space intervals proper to the tiniest and most rapidly-acting elements of technical devices the grains of microstructure and the quanta of microprocesses are so vastly numerous that the matter can be regarded as a continuum with sufficient accuracy. In this approach the matter forms the spatially-continuous medium being subjected to time-continuous processes.

There is, however, an essential distinction between the mathematical and the phenomenologically-physical definition of continuity. The first one is based on the concept of limit at the increments of function and variable both tending to zero. The second one is conditioned by the existence of the apparent limit (symbolized

by $\lim f = \text{limes fictus}$) connected with the finite intervals of the phenomenological averaging over time and space. The intervals mentioned should be sufficiently small to preserve the macroscopic changes but large enough to blur the microscopic variability due to the grain and quantum structure of matter.

Consider the average value of a physical quantity calculated either as a sum of discrete portions, or as an integral, in both cases divided by the corresponding time-interval or spatial interval (regarded as the integration interval in the second case). In the course of changing the denominator of the quotient we shall obtain different results of averaging. At large intervals it will be due to variability of considered quantity in the macroscopic scale of time or space. At sufficiently diminishing intervals, the average value will tend to jump more and more because of the micro-variability of fields among the grains of substance, the discrete character of the structure of matter, and the stochastic nature of the quantum phenomena.

In many cases, however, the average value remains constant within a wide range of changes in the averaging interval, and it forms an apparent limit which distinctly divides the domains of the macroscopic and the microscopic scale.

The existence of an apparent limit allows us to consider the quantity to be phenomenologically continuous in the macroscopic scale. The values of the macroscopic physical fields are ascribed to points, though they should be allocated to very small time-space intervals in reality. The finite increments of the field quantities divided by small increments of time, or of the position coordinate, are considered as the time-derivatives, or the differential operators, respectively.

Since the quantities mentioned can be subjected to the same treatment as the mathematically continuous functions, the descr-

description of physical phenomena is made considerably easier. If only the apparent limit does exist, this idealization does not imply any practical errors.

2.2. The Criteria for Existence of the Apparent Limit

The criteria for existence of the apparent limit may be expressed in terms of times and dimensions characteristic of the macroscopic field on one hand, and of the microstructure and microprocesses of matter on the other.

The macroscopic variability of the field is described by the characteristic time Θ of increase of the process, and the characteristic linear dimension L of the spatial distribution.

Fig. 2.1

Consider a macroscopic time-space field of some scalar quantity α . It is shown in Fig. 2.1 in two shapes: as a time-process $\alpha\langle t \rangle$ at a fixed position $\bar{r} = \text{const}$, and as a distribution along some selected position variable, e.g. $\alpha\langle x \rangle$, at a frozen instant $t = \text{const}$. We look for the point of the biggest slope of each curve and compute the largest absolute value of the derivative $|\partial\alpha/\partial t|_{\text{max}}$ or $|\partial\alpha/\partial x|_{\text{max}}$. Then we determine the difference $|\Delta\alpha|$ between the maximum and minimum value of function α in this particular interval of the monotonic growth or decrease of α which contains the biggest slope. The characteristic time Θ and dimension L are defined as

$$(2.1a) \quad \Theta = \frac{|\Delta\alpha|}{|\partial\alpha/\partial t|_{\text{max}}} > 0, \quad (2.1b) \quad L = \frac{|\Delta\alpha|}{|\partial\alpha/\partial x|_{\text{max}}} > 0$$

Fig. 2.1 shows the graphical way of determination of Θ and L as the horizontal side of a right triangle, in which the hypotenuse is tangent to the curve at the point of biggest slope. For several functions it is quite easy to determine Θ and L by means of analytical reasoning, though in some cases the problem cannot be solved in any way. Since it is sufficient to evaluate the order

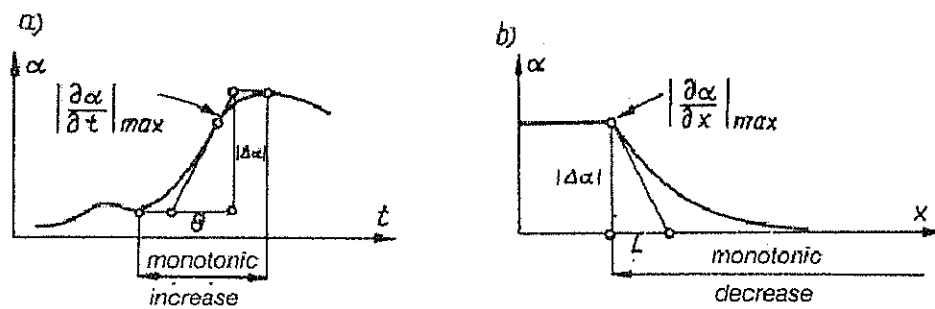


Fig.2.1. Graphic determination of:

- a) the characteristic time θ for transient increase
- b) the characteristic linear dimension L of the macroscopic field quantity α distribution

of Θ and L (but not their exact values), we use the intuition very often. If no other way is available, we take the characteristic linear dimension of considered system for L .

The time-space microvariability is determined by the discrete quantities: the characteristic microtime τ (e.g. the averaged intra-collision time), and the characteristic ^{linear} microdimension λ (e.g. the averaged free path of molecules, the intra-atomic distance).

If the phenomenological intervals of averaging in time δt and in linear dimension δl fulfil the inequalities

$$(2.2a) \quad \tau \ll \delta t \ll \Theta,$$

$$(2.2b) \quad \lambda \ll \delta l \ll L$$

then the considered physical quantity can be macroscopically described by means of a continuous time-space field. In such circumstances the intervals δt and δl secure the disappearance of the discreteness of matter, the macrovariability being still preserved.

The inequalities (2.2) are satisfied in most problems of phenomenological physics and technology but not in all of them. For example a considerably rarefied gas cannot be treated as a continuous medium as long as the mean free path of the molecules is of the same order as the dimensions of the vessel.

There are cases when the time-space field remains phenomenologically continuous with the exception of isolated instants and places in which conditions (2.2) are not fulfilled. In such cases we consider the field to be phenomenologically discontinuous at some distinct surfaces (e.g. the free surface between the liquid and the gas), and at definite instants (e.g. the transition of the shock wave through a fixed point).

2.3. Averaging the Densities of Discrete Sets

Among the extensive quantities, i.e. the quantities seated in a geometric region, we distinguish the substantial quantities

allocated to all the grains of substance, and the componential quantities assigned to the grains of a definite components of a mixture (the classification of quantities will be discussed in detail in Secs. 4.1, 5.2, 6.5). The substantial or componential quantity contained within volume V with n grains is a sum of discrete set of n portions allocated to individual grains.

Particularly, if m_ν is the mass of the ν -th grain taken from the set of all grains, and $m_{i\nu}$ is the mass of the $i\nu$ -th grain of i -th component in the mixture, then the total mass m of all the grains, and the total mass m_i of all the i -th grains, are, respectively:

$$(2.3a) \quad m = \sum_{\nu}^n m_{\nu}, \quad (2.3b) \quad m_i = \sum_{i\nu}^{n_i} m_{i\nu}$$

Dividing the substantial quantity (as well as the componential one) contained within a region by the volume V or mass m , we obtain the average volume-density or mass-density, respectively. The quotient of i -th componential quantity by mass m_i gives the average partial density.

The apparent limits of the densities mentioned become the field quantities in the macroscopic meaning. We need only to take into consideration that the linear interval of phenomenological averaging δl corresponds to a small volume $\delta^3 V$ (of the order of δl^3), a small mass $\delta^3 m$ of all the grains, and a small mass $\delta^3 m_i$ of grains of i -th species. Therefore we are allowed to apply the symbol limf to any quotient which satisfies the condition (2.2b) indirectly:

$$(2.4a) \quad \text{The volume-density} = \lim_{V \rightarrow \delta^3 V} \frac{\sum \dots}{V} = \text{limf} \frac{\sum \dots}{V}$$

$$(2.4b) \quad \text{The mass-density} = \lim_{m \rightarrow \delta^3 m} \frac{\sum \dots}{m} = \text{limf} \frac{\sum \dots}{m}$$

$$(2.4c) \quad \text{The partial density} = \lim_{m_i \rightarrow \delta^3 m_i} \frac{\sum \dots}{m_i} = \text{limf} \frac{\sum \dots}{m_i}$$

For example the (volume-) densities of total mass ϱ , and of mass of i -th species in the mixture ϱ_i , are the respective limfs:

$$(2.5a) \quad \varrho = \limf \frac{\sum m_v}{V}, \quad (2.5b) \quad \varrho_i = \limf \frac{\sum m_{iv}}{V}$$

Fig. 2.2

Fig. 2.2 shows how the field density ϱ is obtained from the average density $\sum m_v/V$, in particular case of a gas flow along the conical channel (Fig. a). The averaging region is placed in the axis of channel. It has the shape of rectangular parallelepiped of constant cross-section area $\S l^2$, variable length x , and volume $V = \S l^2 x$. The average density depends on the position and volume of the region. A large value of x , of the same order as the characteristic dimension L , does not allow to reveal the macroscopic variability of ϱ (Fig. b shows the average densities as the heights of steps). A small value of x , of the same order as the free path of molecules λ , or smaller (possible to be exhibited with the help of the logarithmic scale only, Fig. c), reveals the influence of the discrete structure of matter. It is manifested in the random jumps, according as the region is empty or contains some few grains. In the intermediate interval of x (of the width of a few decades), the average density reaches its apparent limit equal to ϱ .

2.4. The Barycentric and Componential Velocities

Assume that each v -th grain with mass m_v has a definite position \vec{r}_v and velocity \vec{u}_v at the instant given. The discrete set of grains in some region may be considered as a system with mass m determined by Eq.(2.3a), and momentum equal to $\sum m_v \vec{u}_v$. Both quantities are located at the instantaneous centre of system's mass, having the position $\sum m_v \vec{r}_v / m$. The apparent limit of the mass-density (Eq. 2.4b) of momentum is called the barycentric velocity, or velocity of substance \vec{u} :

$$(2.6a) \quad \vec{u} = \limf \frac{\sum m_v \vec{u}_v}{m}$$

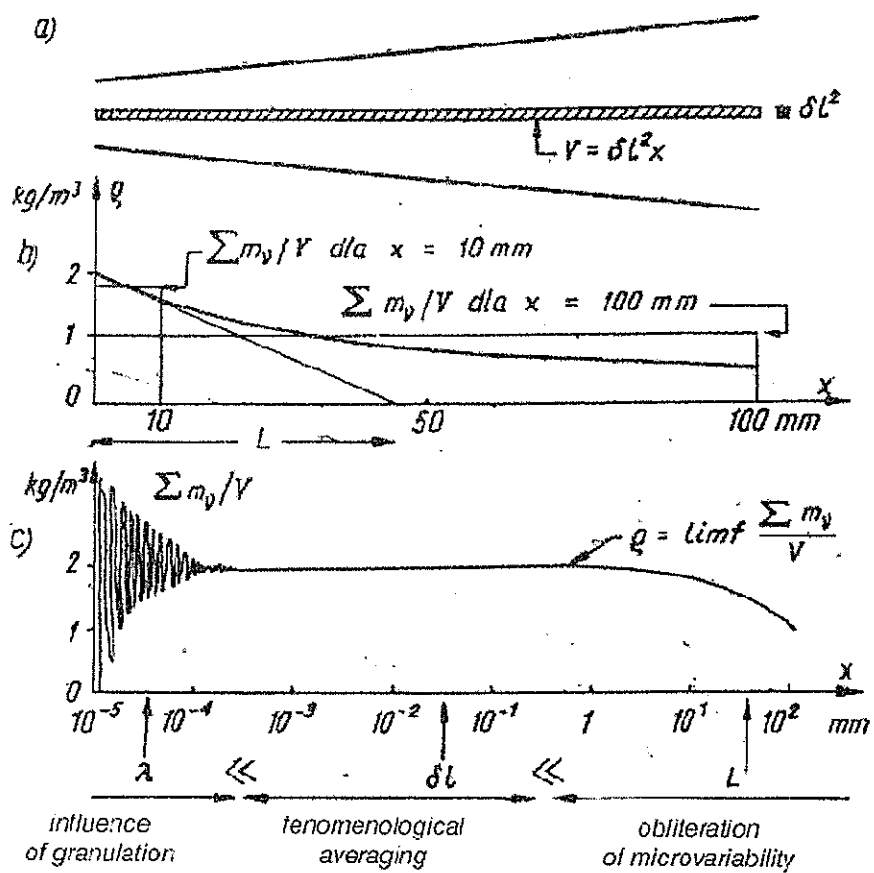


Fig. 2.2. Averaging mass density of gas flowing along the cone-shaped channel

Next, consider the same region but taking into account the grains of i -th species only, with masses m_{iy} and velocities \bar{u}_{iy} . This time the set of grains will have mass m_i determined by Eq. (2.3b), and momentum equal to $\sum m_{iy} \bar{u}_{iy}$, both quantities being located at the instantaneous centre of componential mass. The apparent limit of the partial density (Eq. 2.4c) of component's momentum is called the componential velocity \bar{u}_i :

$$(2.6b) \quad \bar{u}_i = \lim_{m_i} \frac{\sum m_{iy} \bar{u}_{iy}}{m_i}$$

The velocities \bar{u} and \bar{u}_i are field quantities which determine the motion of local centres of mass. In the fluid mechanics, \bar{u} and \bar{u}_i are meant as the velocities of a very small element of fluid or fluid's component, respectively. In reality, such an element does not contain the same grains all the time but it exchanges them with the neighbouring elements, because of the disordered motion of molecules. Hence, with the lapse of time, the same local centre of mass moving with velocity \bar{u} or \bar{u}_i refers to a different grain set every now and again.

The individual grain velocity \bar{u}_y less the barycentric velocity \bar{u} gives grain's individual thermal velocity \bar{u}_y^* , and \bar{u}_y^* multiplied by m_y is called the individual thermal momentum:

$$(2.7a) \quad \bar{u}_y^* = \bar{u}_y - \bar{u}, \quad (2.7b) \quad m_y \bar{u}_y^* = m_y \bar{u}_y - m_y \bar{u}$$

The thermal motion is a relative motion with respect to local centre of mass. At not too low temperature and moderate flow, the thermal motion predominates the macroscopic one, because on an average $|\bar{u}_y| \gg |\bar{u}|$.

Summing up Eq.(2.7b) over the set of grains within region Δ^3V , and using Eqs.(2.6a), (2.3a), we obtain: $\sum m_y \bar{u}_y = m\bar{u}$, $(\sum m_y) \bar{u} = m\bar{u}$, and

$$(2.7c) \quad \sum m_y \bar{u}_y^* = 0$$

The vector-wise addition of thermal momenta of grains in a small region results in their mutual compensation to zero. Thus the thermal motion gives no contribution to phenomenological momentum.

The case of the kinetic energy looks different. According to Eq.(2.7a) the kinetic energy of the ν -th grain is

$$(2.8a) \quad \frac{1}{2} m_\nu (\bar{u}_\nu^2) = \frac{1}{2} m_\nu (\bar{u}^2) + \bar{m}_\nu (\bar{u} \bar{u}_\nu^*) + \frac{1}{2} m_\nu (\bar{u}_\nu^{*2})$$

Summing up Eq.(2.8a) over the set of grains within region $\delta^3 V$, and using Eqs.(2.7c), (2.3a), we obtain: $(\bar{u} \sum m_\nu \bar{u}_\nu^*) = 0$, and

$$(2.8b) \quad \underbrace{\sum \frac{1}{2} m_\nu (\bar{u}_\nu^2)}_{\text{kinetic energy of set of grains}} = \underbrace{\frac{1}{2} m (\bar{u}^2)}_{\text{phenomenological kinetic energy}} + \underbrace{\sum \frac{1}{2} m_\nu (\bar{u}_\nu^{*2})}_{\text{thermal energy}}$$

Thus the kinetic energy of the set of grains is the sum of phenomenological kinetic energy (due to the macroscopic motion), and thermal energy (due to the microscopic motion, with compensated momentum).

2.5. Smoothing the Processes and Distributions.

The phenomenological averaging may be applied to continuous (but not necessarily differentiable) fields that reveal strong variability within intervals of the orders of microtimes τ and microdimensions λ , due to the grain structure of matter and the processes occurring therein.

The time- and space-smoothing method lies in the averaging by means of integration of the actually instantaneous or local field α_x . The integral averaging of a process $\alpha_x \langle t \rangle$ takes place within the time interval Δt , at constant position. The integral averaging of a linear distribution $\alpha_x \langle x \rangle$ takes place within the interval Δx of the position variable, at constant time. If the conditions (2.2) are satisfied, the integral average value reaches its apparent limit, called the smoothed process $\alpha \langle t \rangle$, or the smoothed distribution $\alpha \langle x \rangle$:

$$(2.9a) \quad \alpha\langle t \rangle = \lim_{\Delta t \rightarrow \delta t} \frac{1}{\Delta t} \int_{\Delta t} \alpha_*(\langle t \rangle) dt = \limf \frac{1}{\Delta t} \int_{\Delta t} \alpha_*(\langle t \rangle) dt$$

$$(2.9b) \quad \alpha\langle x \rangle = \lim_{\Delta x \rightarrow \delta l} \frac{1}{\Delta x} \int_{\Delta x} \alpha_*(\langle x \rangle) dx = \limf \frac{1}{\Delta x} \int_{\Delta x} \alpha_*(\langle x \rangle) dx$$

Fig. 2.3

Since the Fourier transformation may be applied to a process or distribution, the fulfilment of conditions (2.2) resolves itself into the existence of a sufficiently wide gap in the amplitude spectrum as a function of frequency (Fig. 2.3c), or of the wave number. Then the spectrum of the actual field α_* breaks up into two distinct parts. The first one corresponds to the macroscopic (smoothed) field α of weak variability (characterized by the time Θ , or dimension L). The second one is connected with the microfield α^* of strong variability (characterized by the microtime τ , or microdimension λ) (Fig. 2.3a,b). The strongly-variable component may take shape of a regular function, as well as of a stochastic fluctuation.

The microfield α^* may be regarded as an instantaneous or local excess of the actual field α_* over the smoothed macrofield α :

$$(2.10) \quad \alpha^* = \alpha_* - \alpha$$

It results from the fulfilment of condition (2.2) that the phenomenological averaging of the smoothed field α gives simply α . Taking this into account and considering definitions (2.9), we find that the microfield vanishes in the course of phenomenological averaging:

$$(2.11a) \quad \frac{1}{\delta t} \int_{\delta t} \alpha^*(\langle t \rangle) dt = 0, \quad (2.11b) \quad \frac{1}{\delta x} \int_{\delta x} \alpha^*(\langle x \rangle) dx = 0$$

Thus the smoothed process or distribution α preserves the macroscopic variability, blurring the microscopic one.

The time-smoothing method originates from the turbulent flow

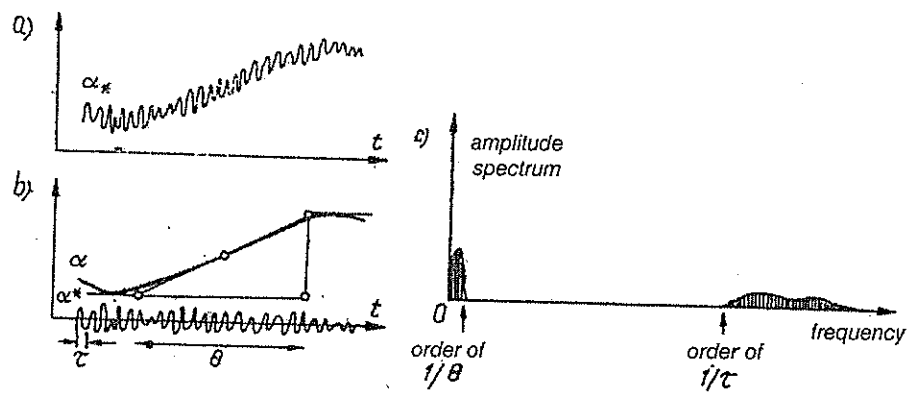


Fig. 2.3. The smoothing of the time run. Temporal real field α_* (a) is a sum of smoothed macro-field α and micro-field α_* (b)

theory: the actual fluid velocity is considered to be a superposition of slow-varying process and fast-varying fluctuation (the time-scale of turbulence, however, is much larger than the time τ characteristic of the intra-molecular collisions). In some astrophysical phenomena the spectrum of electromagnetic radiation may be split up into two distinct parts, due to the radio-waves of considerable length, and relatively short-wave radiation of thermal, visual, and ultraviolet types.

2.6. The Force Fields of Short and Long Range

The forces exerted on the substance are due to influence of some vector fields (e.g. of the gravitational and electromagnetic types) on the masses and charges of the grains.

Since each atom contains both the positive and negative charges, the local structure of electromagnetic field inside of the grains and in their nearest surroundings is extremely intricate. In the dense substance (the solids and liquids), the local electric field vector alternates its direction several times within the intra-atomic or intra-molecular distance λ . If each grain of the body is electrically neutral (i.e. does not contain any surplus charge of definite sign), the electric field generated by its electrons and protons has practical range of the order of λ , and therefore is called the short-range field. Irrespective of this, the substance may be influenced by the long range electric field, its sources being located in the conglomerations of surplus charge. It is proper to remark that the gravitational field carries purely long-range influence, because the mass bears the positive sign only.

According to considerations in Sec. 2.5, the actual electric field \bar{E}_χ is a superposition of the macrofield \bar{E} and microfield \bar{E}^* :

$$(2.12) \quad \bar{E}_\chi = \bar{E} + \bar{E}^*$$

The phenomenological averaging of the field \bar{E}_χ within volume

δ^3V (of the order of δl^3) gives the macroscopic component \bar{E} only, because the mean \bar{E}^* vanishes (see Eq.2.11b):

$$(2.13a) \quad \frac{1}{\delta^3V} \iiint_{\delta^3V} \bar{E}_* d^3V = \bar{E}, \quad (2.13b) \quad \frac{1}{\delta^3V} \iiint_{\delta^3V} \bar{E}^* d^3V = 0$$

We shall consider the smoothed field \bar{E} to be electric field in the phenomenological meaning.

Even in the absence of the macroscopic field \bar{E} , strong electric microfields appear within the substance, at least inside of the grains. The short-range field \bar{E}^* binds the neighbouring atoms in the solid body, making it cohesive and resistant to strain.

Though the average microfield \bar{E}^* vanishes (in the same way as the resultant thermal momentum, Eq.2.7b), nevertheless the force of the grain bonds remains. This kind of interaction is phenomenologically described by the stress tensor, usually with no mention to its very (electric) nature.

The expression for the actual (volume-) density of electric energy $\frac{1}{2}(\bar{E}_*^2)$ may be developed as follows, according to Eq.(2.12):

$$(2.14a) \quad \frac{1}{2}(\bar{E}_*^2) = \frac{1}{2}(\bar{E}^2) + (\bar{E} \bar{E}^*) + \frac{1}{2}(\bar{E}^{*2})$$

Now we integrate the above expression over the phenomenological averaging region δ^3V , where \bar{E} remains constant. Recording that the integral of \bar{E}^* vanishes (Eq.2.13b), we obtain:

$$(2.14b) \quad \iiint_{\delta^3V} \frac{1}{2}(\bar{E}_*^2) d^3V = \frac{1}{2}(\bar{E}^2) \delta^3V + \iiint_{\delta^3V} \frac{1}{2}(\bar{E}^{*2}) d^3V$$

The left-hand side is the electric energy of the actual field \bar{E}_* stored within δ^3V . It consists of two parts connected with the smoothed macroscopic field \bar{E} and the microfield \bar{E}^* , respectively. In the phenomenological meaning, only the first term is regarded as the electric energy. Since it is connected with the bonds of substance, we include the second term in the internal energy.

Eq.(2.14b) is valid for the bodies not subjected to electric polarization. Generally, the density of phenomenological electric energy (i.e. energy of the long-range field) is equal to $\frac{1}{2}(\bar{\mathbf{E}} \bar{\mathbf{D}})$, where $\bar{\mathbf{D}}$ denotes the electric displacement vector (see Eq.4.38).

2.7. Phenomenological Classification of Energies

From the viewpoint of microphysics, the total energy of a system consists of energy of motions of all the substance grains within this system, and energy of all the force fields within the system. While the energy of motions depends on the velocities and masses located within the system only, the field energy is influenced by the distributions of mass, charge, and electric current both inside and outside of the system. Thus the substance is the site of the motion energy, while the field energy may be stored in any place where the field itself does exist, even in the vacuum.

Strictly speaking, to the energies mentioned we have to add the rest energy of mass contained within the system. This term, however, is practically constant as long as the grain velocities remain negligibly small compared to the velocity of light. Since the following reasoning will be confined to the nonrelativistic conditions, the rest energy problem is out of our consideration.

In the phenomenological approach it is convenient to divide the energy of the system into two categories. The first one, called the external energy, is attributed to macroscopic phenomena and depends on the phenomenologically averaged quantities (the barycentric velocity, the long-range field intensities). The second one, called the internal energy, is connected with the microscopic phenomena (the thermal motion, the short-range interaction).

The following table displays two ways of classification of the energies:

(2.15) The Classification of Energies

Phenomenological classification Microphysical classification	External energy (macroscopic phenomena)	Internal energy (microscopic phenomena)
Motion energy	(phenomenological) Kinetic energy (macroscopic motion)	Thermal energy (thermal motion)
Field energy	Gravitational energy (long-range field) (phenomenological) Electromagnetic energy (long-range fields)	Bond energy (short-range fields)

The external energy consists of three distinct components:

- (1) the energy of macroscopic motion of substance, i.e. the kinetic energy (in the phenomenological meaning),
- (2) the energy of gravitational field (which is of long-range nature), i.e. the gravitational energy,
- (3) the energy of long-range electromagnetic fields, i.e. the electromagnetic energy (in the phenomenological meaning).

According to the phenomenological classification, the fourth and the last kind of energy is the internal energy. It consists of the thermal energy due to thermal motion of substance, and the bond energy attributed to short-range fields.

In the multicomponent mixture two kinds of motion contribute no resultant momentum: the ordered motion of grains belonging to particular components in the process of diffusion, and the disordered motion of all the grains of mixture, at statistical equipartition of all directions in space. The diffusion energy (see Sec. 6.10) is included in the thermal energy, as well as the energy of disordered motion.

To the bond energy belong: the energy of intra-molecular bonds in the fluid and intra-atomic bonds in the solid (the elastic

energy), the energy of internal molecular and atomic bonds (the chemical and ionization energies), and the energy of internal nuclear bonds (the nuclear energy).

Since it is hard to separate the shares of thermal and bond energies in many cases, the phenomenological thermodynamics deals with the internal energy as a whole, considering, however, its increments due to various cases.

Since the grains are the carriers of thermal energy, and the short-range fields occur in the grain-filled regions only, we are allowed to make a phenomenological assumption that all the internal energy is located in the substance.

The same kind of location (in the mass of substance) may be accepted for the gravitational energy in terrestrial conditions (see Sec. 5.11). In such circumstances the sum of kinetic energy and gravitational energy may be called the mechanical energy (see Sec. 5.11).

From all those considerations we conclude that among four kinds of phenomenological energies the three (kinetic, gravitational, and internal) are substantial quantities, whereas one (electromagnetic) is a non-substantial quantity.

In the given classification the radiant energy (of electromagnetic radiation) has been omitted. Since from the phenomenological viewpoint it cannot be stored, the radiant energy is considered as a kind of transfer of electromagnetic energy (the radio-wave radiation) and internal energy (the thermal radiation).

3. The Referential Quantities

3.1. Physical Quantities and Motion

There are physical quantities defined or experimentally measured with respect to some frames or objects, and depending on the motion of their references. Imagine we use a mobile point-size gauge or probe in order to examine the physical fields in space at any velocity, and let perform the following ^{thought}~~gedanken~~ experiments on the basis of real observations.

If two infinitesimal gauges could cross the same point at the same time instant but with different velocities, they would give different results of measurement of the electric field intensity.

It has been found long time ago that the accelerations of the same body with respect to two reference systems being in relative rotary motion are different.

In the stationary flow through a nozzle, the immobile probe shows constant fluid velocity, while the probe convected with the stream records time-variability of velocity at the instant when both point-probes cover each other.

If at the same instant a substance element and an acoustic wave-front are passing some immobile point with different velocities, then the differentiating device may record three different values of the time-derivative of pressure, depending on whether the probe rests, is convected with the substance, or "flows" with the wave.

A flow-meter being capable of moving along the pipe shows readings dependent on its velocity with respect to the fluid, particularly zero when it is convected with the stream.

Last of all examples, imagine an examination of the phenomenon of diffusion in the multicomponent mixture by means of the point-gauges convected with the streams of particular components. If the gauges could cross the same point at the same instant, each of them would show a different value of the total mass (of all the componer

in the mixture) flown through unit area per unit time.

From the given examples we see that some physical quantities are influenced by motion, which is not necessarily the motion of substance. Therefore it seems purposeful, especially when dealing with a balance problem, to use an abstractive motion of reference points or systems at the first stage, and substitute some physically specific motion (e.g. the flow of substance or its components, the wave-motion) at a later stage of reasoning.

3.2. Definitions and Denotations of Referential Quantities

The quantities which describe the generalized motion and the quantities depending on the generalized motion are called the referential quantities. The name has been introduced by Rutkowski in 1962, in order to manifest "the reference of a quantity to an abstractive state of motion".

The basic quantity which describes the generalized motion is the referential velocity \bar{w} . This is either the velocity of a selected reference point in space, being called the referential point, or the velocity of a continuous set of referential points forming the referential region. In the latter case the vector \bar{w} forms a time-space field of referential velocity, which determines the displacement, deformation, and dilatation of the referential region per unit time.

The referential point and region are universally-abstract objects, and, generally speaking, they need not be identified with the real grains of substance or phenomenologically continuous substance. It is only when we replace \bar{w} with a physically specified velocity that the referential point and region gain definite properties of substantiality, componentiality, etc.

To the geometric referential quantities belong all the geometrical objects (points, lines, surfaces, spatial regions) being in a stat

of generalized motion. The instantaneous and local displacement of such an object is described by the referential velocity field \bar{w} .

A physical referential quantity is characterized by its dependence on the generalized motion, especially on the referential velocity \bar{w} .

Since our considerations are of nonrelativistic character, we shall determine the referential velocity with respect to a certain stiff frame of reference, assumed as absolute. The geometric objects which rest in this system, the physical quantities referred to the objects mentioned, as well as the physical quantities independent of the state of motion, are called the non-referential quantities. Since the state of rest ($\bar{w} = 0$) is a particular case of the state of motion, the referential quantity bears more general character than the non-referential one.

While the non-referential quantities will be denoted conventionally, the referential quantities will be distinguished by an arrow symbol $\rightarrow\bar{w}$ below the proper letter. For example $\rightarrow\bar{w}V$, $\rightarrow\bar{w}S$, $\rightarrow\bar{w}l$ denote the referential spatial region, surface, and line, respectively, $\rightarrow\bar{w}\alpha$, $\rightarrow\bar{w}\bar{A}$ the referential field quantities of scalar and vector character, respectively. In the particular ^(case of) state of rest ($\bar{w} = 0$) the lower arrow symbol is omitted, because the denoted quantity becomes non-referential.

The arrow symbol, though necessary in the general theory, is a little arduous as applied to specific physical equations. Therefore we shall make deviations from the mentioned principle of notation in some cases. Fortunately, many of the physical field quantities do not depend on the motion of reference point, and those are called the referential invariants symbolized by $\text{inv}\langle\bar{w}\rangle$.

For the sake of easier notation of the integrals with respect to the referential geometric objects, we shall omit the symbol $\rightarrow\bar{w}$ under the differential of integration ($d\bar{l}$, $d^2\bar{s}$, $d^3\bar{v}$). The arrow

symbol, however, will be rigorously placed under the sign of integral. The GGO and Stokes theorems are applicable to integrals of both non-referential and referential field quantities, with respect to the geometric objects both immobile and referential, because the integration process takes place at a frozen instant $t = \text{const.}$

3.3. The Substantial Region

From the phenomenological viewpoint, the substance is continuously distributed in space (with possible existence of certain surfaces of discontinuity of the mass density) and moves according to the barycentric velocity field \bar{u} . The geometric objects containing always the same (phenomenological) elements of substance are called the substantial region $V_{\rightarrow \bar{u}}$, substantial surface $s_{\rightarrow \bar{u}}$, and substantial line $l_{\rightarrow \bar{u}}$.

Each of the mentioned objects ^{is formed of,} ~~forms~~ a set of substantial points which are the instantaneous and local centres of mass within very small regions of the size determined by the apparent limit (see Sec. 2.4). The substantial points must not be identified with the grains of substance. Only in the solid body the substantial points are connected with the structure of the material. In the fluid, the close surroundings of a substantial point contains different molecules every now and again, being continually exchanged with the outlying surroundings.

It results from the definition of barycentric velocity \bar{u} that there is no phenomenological flow of mass through any substantial surface $s_{\rightarrow \bar{u}}$. However, such a surface may be penetrated by the individual grains in their disordered motions, as well as by the different species of grains in their ordered motions (the phenomenon of diffusion). The substantial region contains a constant amount of the grain mass, though the identity of the grains need not be maintained with the lapse of time.

The shell of a substantial region is the substantial surface in its entirety. If a region is surrounded by shell composed of substantial surface and another surface devoid of such a property, then it is not a substantial region. In the thermodynamics, the substantial region is called the closed system (closed for the flow of mass), and the region with partly non-substantial shell - the open system (e.g. a segment of pipe).

The physically-substantial region has its entire shell in the form of a discontinuity surface of the mass density (e.g. the wall of tank with a fluid, the interface between two different solids or non-miscible liquids, the free surface of a non-vaporizing liquid).

On the shell of the imaginary substantial region the mass density field is continuous everywhere. It is a region "cut-out of the space", which does not contain, intersect and touch any interface of phases of substance.

3.4. The Componential Region

The fluid substance may form a mixture of differently moving components. In the phenomenological approach we assume that any i -th component is continuously distributed in space and moves according to the componential velocity field \vec{u}_i . The instantaneous and local centres of i -th mass are considered as the i -th componential points. The componential region $V_{\vec{u}_i}$ and componential surface $S_{\vec{u}_i}$ are the sets of the points mentioned.

It results from the definition of componential velocity \vec{u}_i that there is no phenomenological flow of i -th mass through any componential surface $S_{\vec{u}_i}$. However, such a surface may be penetrated by the i -th grains, as well as the grains of any other species.

There are as many componential regions as the components of mixture. Since the grains of different species are thoroughly mixed,

all the componential regions occupy the same place in space, penetrate each other, and penetrate the sole substantial region which contains the mixture as a whole.

Fig. 3.1 At any point of space the velocities \bar{u}_i of particular components are, generally speaking, different. Thus each componential region displaces in a different way. If at an instant two componential surfaces $s_{\bar{u}_i}$ and $s_{\bar{u}_j}$ cover each other precisely, then at the next instant they will be separated (Fig. 3.1).

3.5. The Non-Substantial Regions

There are spatial regions and surfaces with distinct physical properties or boundaries, nevertheless moving otherwise than the substance. Since the substance may flow across their boundaries, they are called the physical non-substantial regions.

If we are able to determine the phase in a wave of electromagnetic, acoustic, or elastic character, then some specific velocity can be assigned to each point of the region. Such a wave-region is composed of the continuous set of constant-phase wave-surfaces which move with the wave velocity \bar{c} , differing from the velocity of substance \bar{u} .

Because of the inflow of reactants and outflow of products, the combustion zone is a kind of non-substantial region. In case of the detonation combustion, this zone reduces to a very thin layer. From the phenomenological viewpoint, it is a non-substantial discontinuity surface moving with the velocity of the detonation wave. The same model can be applied to the shock-wave surface, the shell of a vapour-bubble in the boiling liquid, or the shell of a fuel droplet vaporizing in hot-gas stream.

3.6. The Referential Differential and Derivative

When a physical quantity is associated, as far as its location is concerned, with some referential point or geometric object, it

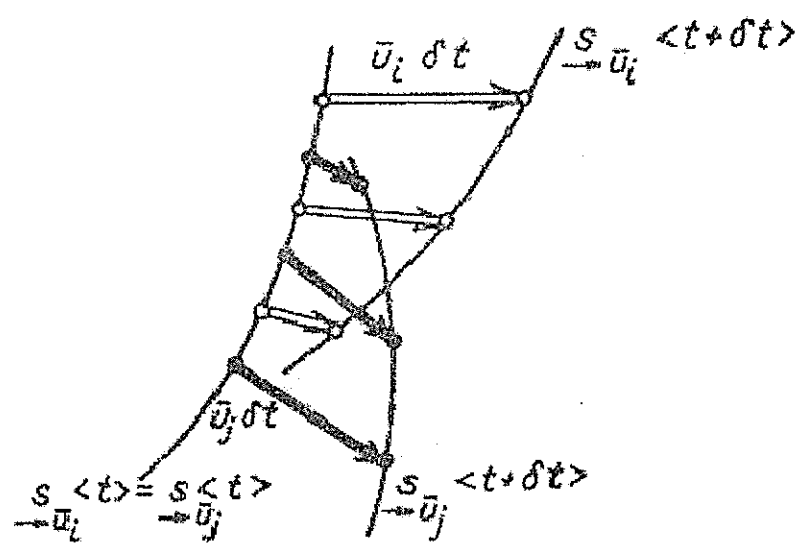


Fig. 3.1. Displacements in time δt of componential surfaces $S_{\bar{u}i}$ and $S_{\bar{u}j}$ which were in line at the time t

eventually becomes a function of time t only, and its increment takes shape of a total differential which includes the indirect influence of the position variability in time.

The total differential of a physical quantity along the displacement (or displacement field) $\bar{w} dt$ made at the referential velocity \bar{w} is called the referential differential, and denoted by $d\ldots\rightarrow\bar{w}$.

The total derivative of a physical quantity determining the time-variability at the referential velocity (or velocity field) \bar{w} is called the referential derivative, and denoted by $\frac{d\ldots}{dt}\rightarrow\bar{w}$.

Both the field and non-field quantities may have referential differentials and derivatives. In case of a point-located quantity, the discussed notions describe the variability at the velocity \bar{w} of the reference point. In case of an integral quantity, the variability is connected with the velocity field \bar{w} existing within a geometric object of reference (the derivative must have the same velocity suffix as that of the region of integration).

While the integral of a non-referential field quantity over a referential region remains the non-referential quantity (the integration takes place at the frozen time instant), the time-derivative of such an integral becomes a referential quantity because of the time-variability of the integration limits. Within the present exposition we shall deal with the referential derivatives of non-referential quantities only.

3.7. The Substantial, Componential, Undulatory, and Local Derivatives

The referential derivative is of universal character. Substituting some specific physical velocities for \bar{w} we obtain particular categories of total derivatives:

Symbol and name of derivative		Reference velocity
(3.1a) $\frac{d}{dt} \rightarrow \bar{w}$	referential	referential (arbitrary) \bar{w}
(3.1b) $\frac{d}{dt} \rightarrow \bar{u}$	substantial	barycentric \bar{u}
(3.1c) $\frac{d}{dt} \rightarrow \bar{u}_i$	componential	componential \bar{u}_i
(3.1d) $\frac{d}{dt} \rightarrow \bar{c}$	undulatory	wave velocity \bar{c}
(3.1e) $\frac{d}{dt} \rightarrow \partial = \frac{\partial}{\partial t}$	local	immobility (0)

The purpose of distinguishing various categories of derivatives of the same quantity may be illustrated by a suggestive though not too precise example (the idea borrowed from Bird, Stewart & Lightfoot, see Bibliography 3).

Imagine that we are measuring the time rate of change of the fish concentration in shallow coastal waters subjected to currents and waving. When the motorboat equipped with the instrument sails with the help of its own propulsion at any arbitrary speed and course, the meter reading corresponds to the referential derivative. When the boat lies adrift, we obtain the substantial derivative. If the fish shoal were dense enough to "carry" the boat, the instrument would show the componential derivative (with respect to fish as a component of the mixture composed of water and fish). When at some speed and course the motorboat remains always at the wave-crest, we obtain the undulatory derivative. When the boat is anchored, the instrument shows the local derivative.

The arrow-type denotation of derivatives is necessary in the general theory and in some specific physical problems which require to distinguish various rates of change of the same quantity. In the fluid mechanics and nonequilibrium thermodynamics, the sub-

stantial derivative is frequently denoted by $\frac{D}{Dt}$, the customary symbol of the total derivative $\frac{d}{dt}$ being used as well. This kind of notation, however, disregards the existence of any other velocity than the flow velocity.

3.8. Referential Derivative of a Field Quantity

The physical field quantities, e.g. the scalars α , β , γ , are characterized by two kinds of functional relations: the physical dependence of one quantity on the others, e.g. the state equation of the type $\alpha(\beta, \gamma)$, and the time-space dependence $\alpha(t, \vec{r})$ on time t and position \vec{r} .

Fig. 3.2

Consider at first a simple example of a scalar field α which is function of time t , and of the only position variable x (Fig. 3.2). At the earlier and later instants, t' and t'' , respectively, differing by a small time interval $\delta t = t'' - t' > 0$, the field α has two different spatial distributions $\alpha(t', x)$ and $\alpha(t'', x)$. Assume that the referential point is moving with some velocity, the x -th component of which being w_x , and attains the coordinate values x' and x'' at the instants t' and t'' , respectively. The difference in the coordinate values is the referential displacement $x'' - x' = \delta x = w_x \delta t$.

It results from both analytic and graphic reasoning (Fig. 3.2) that the referential increment of α equal to $\delta \alpha_{\vec{w}} = \alpha(t'', x'') - \alpha(t', x')$ can be decomposed into two parts. The first one is the increment occurred during time δt at constant position $x = x'$, and is approximately equal to $\alpha(t'', x') - \alpha(t', x') = \frac{\partial \alpha}{\partial t} \delta t$. The second one is the increment occurred along the displacement δx at constant time $t = t''$, being approximately equal to $\alpha(t'', x'') - \alpha(t'', x') = \frac{\partial \alpha}{\partial x} \delta x = w_x \frac{\partial \alpha}{\partial x} \delta t$.

Now we divide the referential increment consisting of two parts by δt , approach the limit ($\delta \rightarrow 0$), and obtain the referential derivative of a scalar one-dimensional field:

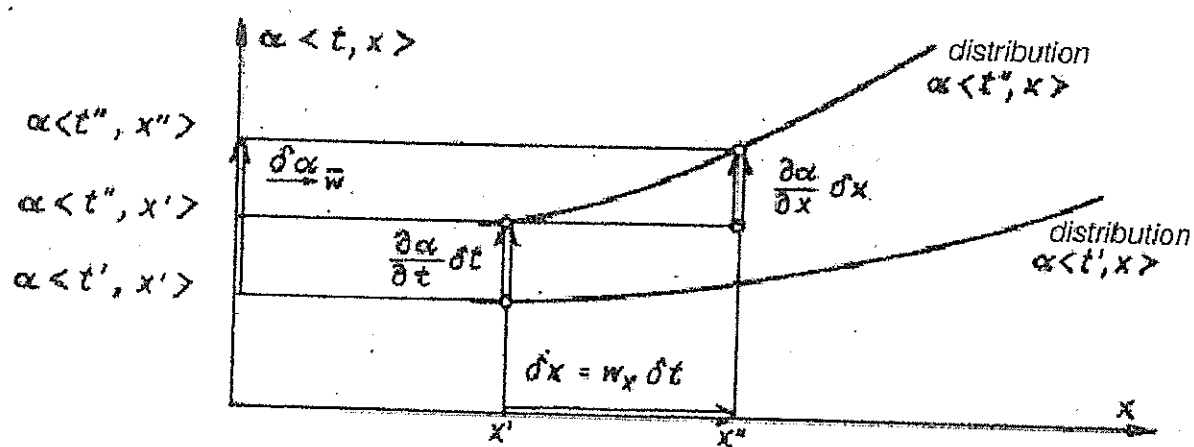


Fig. 3.2. Referential increment of one-dimensional field α

In general, the referential derivative of the spatial differential operator (gradient, divergence, curl) is unequal to the operator of the referential derivative, save for the case of $\bar{w} = 0$, when for example

$$(3.5a) \quad \frac{\partial \operatorname{div} \bar{A}}{\partial t} = \operatorname{div} \frac{\partial \bar{A}}{\partial t}, \quad (3.5b) \quad \frac{\partial \operatorname{rot} \bar{A}}{\partial t} = \operatorname{rot} \frac{\partial \bar{A}}{\partial t}$$

because the succession of partial differentiation with respect to time and position variables may be arbitrary.

3.9. The Stationary and Homogenous Fields. The Referential Strouhal Number

The shares of the first and second term in the decomposed referential derivative (Eq.3.3) depend on the predomination of either time-wise or space-wise variability of the field. One of the terms vanishes in two extreme cases of the stationary field, not varying in time at any immobile point ($\partial / \partial t = 0$), and of the homogenous field of constant distribution (zero gradient) at any instant:

$$\begin{aligned} (3.6a) \quad \frac{d\alpha}{dt} \bar{w} &= (\bar{w} \operatorname{grad} \alpha), & \frac{d\bar{A}}{dt} \bar{w} &= \bar{w} \operatorname{grad} \bar{A} \\ &\text{for stationary field,} & \partial \alpha / \partial t &= 0, \quad \partial \bar{A} / \partial t = 0 \\ (3.6b) \quad \frac{d\alpha}{dt} \bar{w} &= \frac{\partial \alpha}{\partial t}, & \frac{d\bar{A}}{dt} \bar{w} &= \frac{\partial \bar{A}}{\partial t} \\ &\text{for homogenous field,} & \operatorname{grad} \alpha &= 0, \quad \operatorname{grad} \bar{A} = 0 \end{aligned}$$

The only time-change of a stationary field may result from the motion of the referential point, provided the field has variable spatial distribution. In a homogenous field the referential and local derivatives are equal at any velocity \bar{w} .

It results from the definitions of characteristic time of increase Θ (Eq.2.1a) and characteristic linear dimension L (Eq.2.1b) that the order of the ratio of the second and first term in the referential derivative (Eq.3.3) is determined by so-called referential Strouhal number Str_r :

$$(3.7) \quad \left\{ \frac{|\overline{w} \overline{\text{grad}} \alpha|}{\left| \frac{\partial \alpha}{\partial t} \right|} \right\} \text{ is of the order of } \frac{|\overline{w}| \textcircled{H}}{L} = \text{Strr} \geq 0$$

The Strr number obviously depends on the referential velocity \overline{w} (the substantial Strouhal number $\text{Str} = |\overline{u}| \textcircled{H} / L$, known in the fluid mechanics, is but a particular case of Strr at $\overline{w} = \overline{u}$). Comparing Eqs.(3.6) and (3.7) we conclude that, at a given velocity \overline{w} , in case of $\text{Strr} \gg 1$ the field can be regarded as quasi-stationary, while in case of $\text{Strr} \ll 1$ as quasi-homogenous.

lg. 3.3 Exercise 3.1. Show that for a plane non-deforming wave (Fig. 3.3) propagated with constant velocity $|\overline{w}| = |\overline{c}| = c$ we have $\text{Strr} = 1$.

Instructions. The described type of wave can be represented by the one-dimensional field $\alpha(t, x)$ with distributions at the instants t' and $t'' = t' + \delta t$ identical in shape but shifted along the x axis by $\delta x = c \delta t$, so there is no referential increment of the field α at a point moving with velocity c . We use this property in Eq.(3.2a) at $w_x = c$ and compute the ratio of absolute values in Eq.(3.7) which is equal to 1, because for the wave considered it is always $L = c \textcircled{H}$.

Exercise 3.2. Show that in case of a spatial acoustic or elastic wave of small damping and velocity $|\overline{c}|$ much bigger than the velocity of substance $|\overline{u}|$, the acceleration of substance can be approximately expressed as the partial derivative $\partial \overline{u} / \partial t$.

Instructions. The acceleration of substance is the substantial derivative of \overline{u}

$$(3.8a) \quad \frac{d\overline{u}}{dt} \rightarrow \overline{u} = \frac{\partial \overline{u}}{\partial t} + \overline{u} \cdot \overline{\text{grad}} \overline{u}$$

and the undulatory derivative of \overline{u} at small damping (nearly non-deforming wave) approximately equals zero (see Ex. 3.1)

$$(3.8b) \quad \frac{d\overline{u}}{dt} \rightarrow \overline{c} = \frac{\partial \overline{u}}{\partial t} + \overline{c} \cdot \overline{\text{grad}} \overline{u} \cong 0 \quad (\text{Strr} \cong 1 \text{ for } \overline{w} = \overline{c})$$

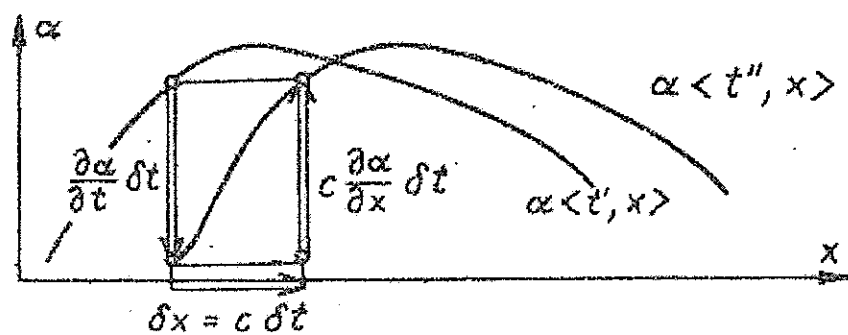


Fig. 3.3. Plane non-deforming wave

We substitute Eq.(3.8b) into Eq.(3.8a), take into account the inequality $|\vec{c}| \gg |\vec{u}|$, use Eq.(3.8b) once again, and finally conclude that the substantial and partial derivatives of \vec{u} are practically equal.

3.10. Referential Derivative of Small Line Element

While being in motion, a referential geometric object may be subjected to deformation. Consider the simplest geometric element which is a very small oriented line segment $\delta\vec{l}$ moving in accordance with the referential velocity field \vec{w} . The rate of change of vector $\delta\vec{l}$ is expressed by $\frac{d\delta\vec{l}}{dt} \rightarrow \vec{w}$. This kind of referential derivative, however, cannot be decomposed with the help of rule (3.3b), because we deal with one selected segment only, which is not a field quantity.

Fig. 3.4 The segment $\delta\vec{l}$ describes the relative position of two close referential points (Fig. 3.4). The beginning point (') and the end point (") of vector $\delta\vec{l}$ are displacing along paths (') and ("), respectively. During time dt both points are subjected to translations $\vec{w}'dt$ and $\vec{w}''dt$. At the instants t and $t + dt$ the mobile segment identifies itself with definitely situated vectors $\delta\vec{l}_I$ and $\delta\vec{l}_{II}$.

If $\delta\vec{l}$ is small enough to neglect the curvature of the displacement trajectories, then it results from the disposition of vectors shown in Fig. 3.4 that

$$(3.9) \quad \delta\vec{l}_{II} - \delta\vec{l}_I = \vec{w}''dt - \vec{w}'dt$$

The left-hand side is the referential increment of the non-field vector $\delta\vec{l}$, i.e. $\frac{d\delta\vec{l}}{dt}$. The increment of the velocity field \vec{w} on the right-hand side occurs along the line element $\delta\vec{l}$, and according to rule (1.37b) is expressed by $\vec{w}'' - \vec{w}' = \overline{\delta\vec{l} \cdot \nabla} \vec{w}$. Taking this into account in Eq.(3.9) and dividing by dt we obtain the referential derivative of line element $\delta\vec{l}$:

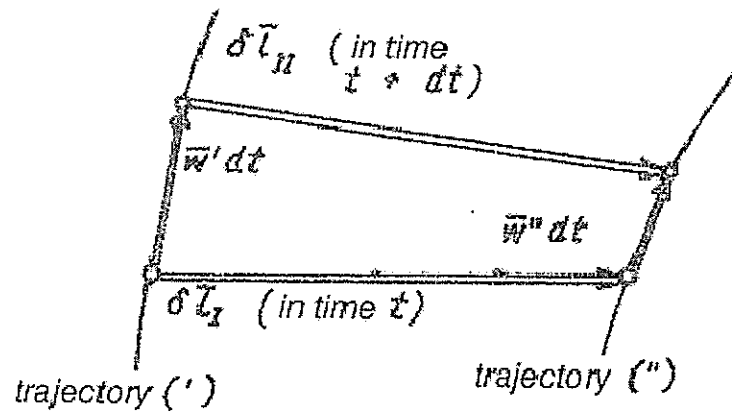


Fig. 3.4. Displacement of the small segment $\delta \bar{l}$ during the time dt

$$(3.10a) \quad \frac{d\delta\bar{l}}{dt} \rightarrow \bar{w} \approx \overline{\delta\bar{l}' \bar{g} \bar{r} \bar{a} \bar{d} \bar{w}}$$

The above relation is approximate. It becomes precise when the small segment $\delta\bar{l}$ approaches the limit of infinitesimal line element $d\bar{l}$.

A similar formula is known in the elasticity theory. When each substantial point of a body is subjected to finite but small deformative displacement $\delta\bar{r}$ with respect to its undistorted shape, then the finite but small increment of the line element $\delta\bar{l}$ is:

$$(3.10b) \quad \delta(\delta\bar{l}) = \overline{\delta\bar{l}' \bar{g} \bar{r} \bar{a} \bar{d} \delta\bar{r}}$$

3.11. Deformation of a Referential Region

It is not only a line but also a surface and spatial region that can be built-up on an array of line elements, which change in time according to Eq.(3.10a) (at $\delta\bar{l} \rightarrow d\bar{l}$). For that reason the deformation of referential region per unit time may be expressed in a most general way by the tensor field of the deformation rate $\bar{g} \bar{r} \bar{a} \bar{d} \bar{w}$. According to rules (1.13b) and (1.29a) it may be decomposed into three terms:

$$(3.11) \quad \bar{g} \bar{r} \bar{a} \bar{d} \bar{w} = \frac{\bar{l}}{3} \text{div } \bar{w} + \bar{g} \bar{r} \bar{a} \bar{d} \text{ds} \bar{w} + \bar{g} \bar{r} \bar{a} \bar{d} \text{a} \bar{w}$$

The first term is a spherical tensor, its scalar trace $\text{Tr}(\frac{\bar{l}}{3} \text{div } \bar{w}) = \text{div } \bar{w}$ (see Eq.1.11b) being called the dilatation rate. It is the time rate of change of sum of the relative elongations of any three line elements $d\bar{l}$, provided they are triply orthogonal at the instant given. On the other hand, it is the rate of relative change of volume d^3V of an elementary cell of the region (see Eq.3.3).

The third term is an antisymmetric tensor. Its pseudo-vector with opposite sign $-\text{Vec } \bar{g} \bar{r} \bar{a} \bar{d} \text{a} \bar{w} = \frac{1}{2} \text{rot } \bar{w}$ (see Eqs.1.29b, 1.12), called the rotation rate, is the average angular velocity of any three line elements $d\bar{l}$, provided they are triply orthogonal at the instant given.

Last of all, the second term with the form of a symmetric deviator $\bar{g}\bar{r}\bar{a}\bar{d}^{ds}\bar{w}$, is called the pure strain rate (or the rate of strain of shape). It describes the time rate of change of the angles between line elements $d\bar{l}$, which are triply orthogonal at the instant given and form an elementary cell.

Fig. 3.5

Leaving the detailed analysis of deformation out, we shall illustrate the process by an example of a plane figure shown in Fig. 3.5. It is a square which has been converted into a rotated and diminished trapezium, at exaggerated changes of angles and lengths of sides. The resultant process of deformation has been decomposed into three phases: the rotation (the square rotated by the angle of $\frac{1}{2} \bar{\text{rot}} \bar{w} dt$, its shape and size being preserved), the pure strain (the square converted into trapezium of the same area and identical axes of symmetry), and the dilatation (identical relative diminution of both orthogonal dimensions by $\frac{1}{3} \text{div} \bar{w} dt$, the shape and angular position being preserved).

In the theory of elasticity, the symmetric deviator of pure strain $\bar{g}\bar{r}\bar{a}\bar{d}^{ds}\bar{r}$ and the scalar of dilatation $\text{div} \bar{r}$ are usually extracted out of the deformation tensor $\bar{g}\bar{r}\bar{a}\bar{d} \bar{r}$.

Some particular properties characterize the incompressible region (e.g. filled with incompressible liquid), which is non-dilating (or keeping its volume unchanged)

(3.12a) $\text{div} \bar{w} = 0$ always and everywhere in incompressible region, and the rigid region (e.g. rigid body), which is submitted neither to dilatation nor the pure strain, but may rotate as a whole in accordance with a homogenous field of angular velocity $\bar{\omega}_0$

(3.12b) $\bar{g}\bar{r}\bar{a}\bar{d}^{s}\bar{w} = 0$, $\bar{g}\bar{r}\bar{a}\bar{d} \bar{w} = \bar{g}\bar{r}\bar{a}\bar{d}^a \bar{w}$, $\bar{\omega}_0 = \frac{1}{2} \bar{\text{rot}} \bar{w} = \text{const} \langle \bar{r} \rangle$

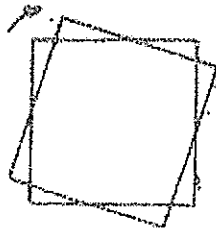
always and everywhere in rigid region

square
in time t

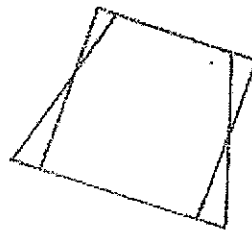


rotated and diminished
trapezium in time $t+dt$

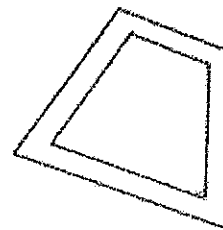
deformation
on displacement $\bar{w} dt$



rotation



pure strain



dilatation

Fig. 3.5. Transformation of a plane figure resulting from the referential displacement of its points

3.12. The Leibniz-Reynolds Transformation

Fig. 3.6 In order to compute the rate of change of a volume integral within mobile limits we have to consider a referential spatial region $V_{\vec{w}}$ and its shell $s_{\vec{w}}$. At two close instants differing by δt the moving region takes two definite positions (Fig. 3.6): $V_{\vec{w}}(t) = V_1$, $s_{\vec{w}}(t) = s_1$, $V_{\vec{w}}(t+\delta t) = V_2$, $s_{\vec{w}}(t+\delta t) = s_2$. While the volumes $V_1, V_2 > 0$ are positive, the increment $\delta V = V_2 - V_1 \gtrless 0$ may be of either positive or negative sign, depending on expansion or contraction of the region.

The change in volume results from the displacement of shell $s_{\vec{w}}$. During time δt it sweeps a thin spatial slice of local thickness $\vec{w} \delta t$. Each surface element $d^2 \vec{s}_1$ forms a base for the volume element of the slice swept (compare Eq.1.23d)

$$(3.13) \quad d^2(\delta V) = (d^2 \vec{s}_1 \vec{w} \delta t) \begin{cases} > 0 \text{ at acute angle} \\ < 0 \text{ at obtuse angle} \end{cases} \text{ between } d^2 \vec{s}_1 \text{ and } \vec{w}$$

The difference of two integrals of a scalar field α , or a vector field \vec{A} , over the regions V_2 and V_1 but at the same instant t is equal to the integral over the slice δV , because the contributions of overlapping parts of V_1 and V_2 cancel each other. Since the thickness of the slice is finite, the mentioned difference is but a double integral over the shell s_1 (Eq.3.13), so we can apply the GGO theorem (1.34), and, in case of the field \vec{A} , the rule (1.20a) in addition:

$$(3.14a) \quad \iiint_{V_2} \alpha(t) d^3 V - \iiint_{V_1} \alpha(t) d^3 V = \iint_{\delta V} \alpha d^2(\delta V) = \delta t \oint_{s_1} (d^2 \vec{s}_1 \vec{w} \alpha) = \\ = \delta t \iiint_{V_1} \text{div}(\vec{w} \alpha) d^3 V$$

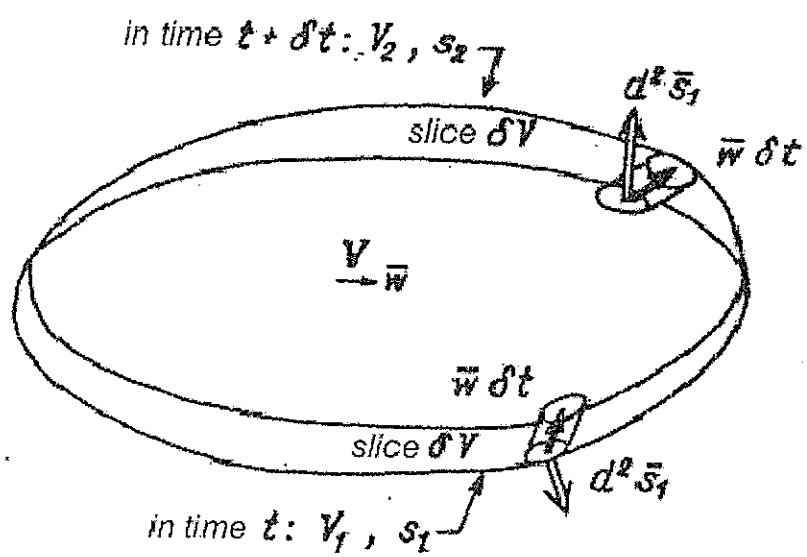


Fig. 3.6. Small displacement of the referential spatial region

$$\begin{aligned}
 (3.14b) \quad \iiint_{V_2} \bar{A} \langle t \rangle d^3V - \iiint_{V_1} \bar{A} \langle t \rangle d^3V &= \iiint_{\delta V} \bar{A} d^2(\delta V) = \delta t \oint_{S_1} (d^2 \bar{s}_1 \bar{w}) \bar{A} = \\
 &= \delta t \oint_{S_1} \overline{d^2 s_1 \bar{w} \bar{A}} = \delta t \iiint_{V_1} \overline{\text{div } \bar{w} \bar{A}} d^3V
 \end{aligned}$$

For a very small δt and at immobile point, we have

$$(3.15a) \quad \alpha \langle t + \delta t \rangle - \alpha \langle t \rangle = \frac{\partial \alpha}{\partial t} \delta t, \quad (3.15b) \quad \bar{A} \langle t + \delta t \rangle - \bar{A} \langle t \rangle = \frac{\partial \bar{A}}{\partial t} \delta t$$

Consequently, we transform the difference of integrals computed at slightly differing instants but over the same region V_2 :

$$(3.16a) \quad \iiint_{V_2} \alpha \langle t + \delta t \rangle d^3V - \iiint_{V_2} \alpha \langle t \rangle d^3V = \delta t \iiint_{V_2} \frac{\partial \alpha}{\partial t} d^3V$$

$$(3.16b) \quad \iiint_{V_2} \bar{A} \langle t + \delta t \rangle d^3V - \iiint_{V_2} \bar{A} \langle t \rangle d^3V = \delta t \iiint_{V_2} \frac{\partial \bar{A}}{\partial t} d^3V$$

Let prefix $\delta_{\vec{w}}$ denote a small referential increment of the volume integral, being the difference in value of integral for the region V_2 at the instant $t + \delta t$, and for the region V_1 at the instant t . Adding and subtracting an identical term we obtain the sum of two differences of integrals: the first one concerns the same region V_2 but two different times, while the second one - different regions at the same instant t . Substituting Eqs. (3.16) and (3.14)

$$\begin{aligned}
 \delta_{\vec{w}} \iiint_{V_2} \alpha d^3V &= \iiint_{V_2} \alpha \langle t + \delta t \rangle d^3V - \iiint_{V_2} \alpha \langle t \rangle d^3V + \iiint_{V_2} \alpha \langle t \rangle d^3V - \iiint_{V_1} \alpha \langle t \rangle d^3V = \\
 &= \delta t \left\{ \iiint_{V_2} \frac{\partial \alpha}{\partial t} d^3V + \iiint_{V_1} \text{div}(\bar{w} \alpha) d^3V \right\}
 \end{aligned}$$

$$\begin{aligned}
 \delta_{\vec{w}} \iiint_{V_2} \bar{A} d^3V &= \iiint_{V_2} \bar{A} \langle t + \delta t \rangle d^3V - \iiint_{V_2} \bar{A} \langle t \rangle d^3V + \iiint_{V_2} \bar{A} \langle t \rangle d^3V - \iiint_{V_1} \bar{A} \langle t \rangle d^3V = \\
 &= \delta t \left\{ \iiint_{V_2} \frac{\partial \bar{A}}{\partial t} d^3V + \iiint_{V_1} \overline{\text{div } \bar{w} \bar{A}} d^3V \right\}
 \end{aligned}$$

then dividing both sides by δt and approaching the limit ($\delta \rightarrow 0$,

$V_1, V_2 \rightarrow \vec{V}_{\vec{w}}$), we obtain the Leibniz-Reynolds transformation:

$$(3.17a) \quad \frac{d}{dt} \int_{\vec{V}_{\vec{w}}} \alpha \, d^3V = \int_{\vec{V}_{\vec{w}}} \left\{ \frac{\partial \alpha}{\partial t} + \text{div}(\vec{w} \alpha) \right\} d^3V \quad \text{for scalar field } \alpha$$

$$(3.17b) \quad \frac{d}{dt} \int_{\vec{V}_{\vec{w}}} \vec{A} \, d^3V = \int_{\vec{V}_{\vec{w}}} \left\{ \frac{\partial \vec{A}}{\partial t} + \overline{\text{div}} \, \vec{w} \cdot \vec{A} \right\} d^3V \quad \text{for vector field } \vec{A}$$

Exercise 3.3. Find such forms of the transformations (3.17) which contain the surface integrals of $(d^2\vec{s} \cdot \vec{w})\alpha$ and $(d^2\vec{s} \cdot \vec{w})\vec{A}$ on the right-hand sides.

Instructions. Apply the GGO theorem (1.34) and rule (1.20a) to the proper parts of the volume integrals on the right-hand sides of Eqs.(3.17a,b).

3.13. The Leibniz-Helmholtz Transformation

Fig. 3.7 In order to compute the rate of change of a surface integral of ~~(a vector field \vec{A} (as a scalar product))~~

within mobile limits we have to consider a referential open surface $\vec{s}_{\vec{w}}$ and its contour $\vec{l}_{\vec{w}}$. At two close instants the moving surface takes two definite positions (Fig. 3.7): $\vec{s}_{\langle t \rangle} = s_1$, $\vec{l}_{\langle t \rangle} = l_1$, $\vec{s}_{\langle t+\delta t \rangle} = s_2$, $\vec{l}_{\langle t+\delta t \rangle} = l_2$.

During time δt the surface $\vec{s}_{\vec{w}}$ sweeps a thin spatial slice δV of local thickness $\vec{w} \, \delta t$. Each surface element $d^2\vec{s}_1$ forms a base for the volume element of the slice swept $d^2(\delta V)$, being defined by Eq.(3.13). At the same time the referential contour $\vec{l}_{\vec{w}}$ sweeps a narrow closed stripe δs_3 of local width $\vec{w} \, \delta t$, bordered by the contours l_1 and l_2 . Each line element $d\vec{l}_1$ forms a base for the surface element of the stripe swept (compare Eq.1.23c)

$$(3.18) \quad d(\delta \vec{s}_3) = \overline{d\vec{l}_1 \times \vec{w}} \, \delta t$$

The succession of factors in the vector product is such that at the right-handed orientation between surface s_1 and its contour l_1 the stripe δs_3 is outwardly bound with respect to slice δV .

Notice that the shell s_0 of the slice δV consists of 3 open

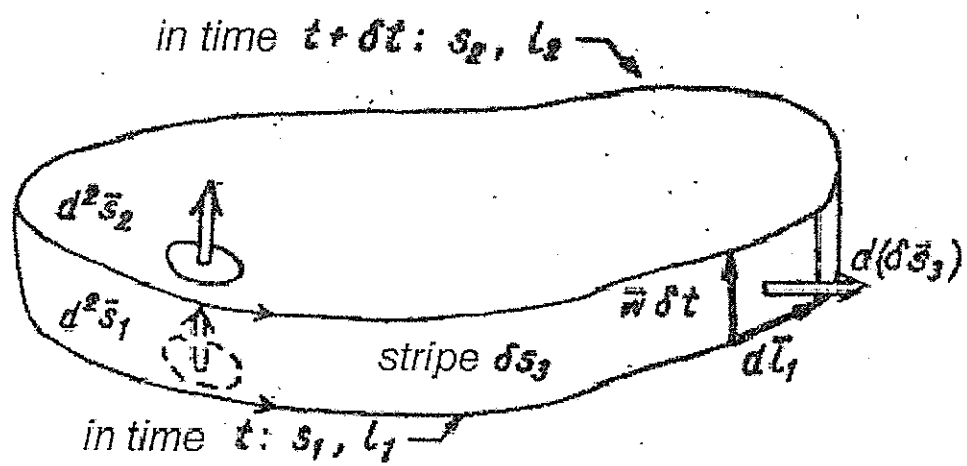


Fig. 3.7. Small displacement of the referential open surface

surfaces s_1 , s_2 , and δs_3 , among which only s_1 is inwardly bound. We denote this with the help of symbolic equation

$$s_0 = (-s_1) + s_2 + \delta s_3, \quad \text{or} \quad s_2 - s_1 = s_0 - \delta s_3.$$

Consequently, the difference between integrals over s_2 and s_1 at the same instant t is equal to the integral over the shell s_0 minus the integral over the stripe δs_3 . To the first one the GGO theorem (1.34a) can be applied. Since the width of stripe is finite, the second one is but a single integral along the contour l_1 , and thus subject to Stokes' theorem (1.35). Taking all this into consideration and using Eqs.(3.18), (3.13), and the rule (1.19a), we obtain

$$\begin{aligned} (3.19) \quad & \iint_{s_2} (\bar{A} \langle t \rangle d^2 \bar{s}) - \iint_{s_1} (\bar{A} \langle t \rangle d^2 \bar{s}) = \oint_{s_0} (\bar{A} d^2 \bar{s}) - \int_{\delta s_3} (\bar{A} d\{\delta \bar{s}_3\}) = \\ & = \iint_{\delta V} \text{div } \bar{A} d^2(\delta V) - \oint_{l_1} (\overline{d\bar{l}_1 \times \bar{w}} \bar{A}) = \delta t \left\{ \iint_{s_1} \text{div } \bar{A} (d^2 \bar{s}_1 \bar{w}) - \oint_{l_1} (d\bar{l}_1 \overline{\bar{w} \times \bar{A}}) \right\} = \\ & = \delta t \iint_{s_1} (\{\bar{w} \text{div } \bar{A} - \overline{\text{rot } \bar{w} \times \bar{A}}\} d^2 \bar{s}) \end{aligned}$$

According to Eq.(3.15b) the difference of integrals computed at slightly differing instants but over the same surface s_2 is

$$(3.20) \quad \iint_{s_2} (\bar{A} \langle t + \delta t \rangle d^2 \bar{s}) - \iint_{s_2} (\bar{A} \langle t \rangle d^2 \bar{s}) = \delta t \iint_{s_2} \left(\frac{\partial \bar{A}}{\partial t} d^2 \bar{s} \right)$$

The referential increment of the surface integral is the difference in value of integral for the surface s_2 at the instant $t + \delta t$, and for the surface s_1 at t . Using the same procedure as in Sec. 3.12, substituting Eqs.(3.20) and (3.19)

$$\begin{aligned} \delta_{\bar{w}} \iint_{s_2} (\bar{A} d^2 \bar{s}) &= \iint_{s_2} (\bar{A} \langle t + \delta t \rangle d^2 \bar{s}) - \iint_{s_2} (\bar{A} \langle t \rangle d^2 \bar{s}) + \iint_{s_2} (\bar{A} \langle t \rangle d^2 \bar{s}) - \iint_{s_1} (\bar{A} \langle t \rangle d^2 \bar{s}) \\ &= \delta t \left\{ \iint_{s_2} \left(\frac{\partial \bar{A}}{\partial t} d^2 \bar{s} \right) + \iint_{s_1} (\{\bar{w} \text{div } \bar{A} - \overline{\text{rot } \bar{w} \times \bar{A}}\} d^2 \bar{s}) \right\} \end{aligned}$$

then dividing both sides by δt , and approaching the limit ($\delta \rightarrow d$, $s_1, s_2 \rightarrow s_{\bar{w}}$), we obtain the Leibniz-Helmholtz transformation:

$$(3.21) \quad \frac{d}{dt} \int_{s_{\bar{w}}} (\bar{A} d^2 \bar{s}) = \int_{s_{\bar{w}}} \left(\left\{ \frac{\partial \bar{A}}{\partial t} + \bar{w} \operatorname{div} \bar{A} - \operatorname{rot} \bar{w} \times \bar{A} \right\} d^2 \bar{s} \right)$$

It is possible to show (Ex. 3.5) that if $s_{\bar{w}}$ is a shell, then

$$(3.22) \quad \frac{d}{dt} \oint_{s_{\bar{w}}} (\bar{A} d^2 \bar{s}) = \oint_{s_{\bar{w}}} \left(\left\{ \frac{\partial \bar{A}}{\partial t} + \bar{w} \operatorname{div} \bar{A} \right\} d^2 \bar{s} \right)$$

Exercise 3.4. Find such a form of transformation (3.21), the right-hand side of which contains the integral of $(\bar{w} \times \bar{A} d\bar{l})$ along the contour $l_{\bar{w}}$.

Instruction. Apply Stokes' theorem (1.35) to the proper part of the surface integral on the right-hand side of Eq.(3.21).

Exercise 3.5. Prove transformation (3.22) in two different ways.

Instructions. In the first method, we use the result of Ex. 3.4 adapted for the case of vanishing contour $l_{\bar{w}}$ (when the open surface becomes a shell). In the second method we apply the GGO theorem (1.34a), transformation (3.17a), and rule (3.5a) to the left-hand side of Eq.(3.22). Using the GGO theorem (1.34a) reversely, we obtain the right-hand side of Eq.(3.22).

3.14. The Leibniz-Thomson Transformation

Leaving out the proof, we present the Leibniz-Thomson transformation of a vector field \bar{A} (as a scalar product) for an integral along the referential open line $l_{\bar{w}}$:

$$(3.23) \quad \frac{d}{dt} \int_{l_{\bar{w}}} (\bar{A} d\bar{l}) = \int_{l_{\bar{w}}} \left(\left\{ \frac{\partial \bar{A}}{\partial t} + \operatorname{grad}(\bar{w} \bar{A}) - \bar{w} \times \operatorname{rot} \bar{A} \right\} d\bar{l} \right)$$

It is possible to show (Ex. 3.7) that if $l_{\bar{w}}$ is a contour, then

$$(3.24) \quad \frac{d}{dt} \oint_{l_{\bar{w}}} (\bar{A} d\bar{l}) = \oint_{l_{\bar{w}}} \left(\left\{ \frac{\partial \bar{A}}{\partial t} - \bar{w} \times \operatorname{rot} \bar{A} \right\} d\bar{l} \right)$$

Exercise 3.6. Find such form of transformation (3.23), the

right-hand side of which contains the difference

$$[(\bar{w} \bar{A})]_2 - [(\bar{w} \bar{A})]_1$$

of values of the field $(\bar{w} \bar{A})$ at the end-point 2 and the beginning-point 1 of the line $\underline{l}_{\bar{w}}$.

Instruction. Apply the rule (1.36a) to the proper part of the integral on the right-hand side of Eq.(3.23).

Exercise 3.7. Prove transformation (3.24) in two different ways.

Instructions. In the first method, we use the result of Ex. 3.6 for the points 1 and 2 covering each other (when the open line becomes a contour). In the second method, we apply Stokes' theorem (1.35), transformation (3.21), and rules (1.30d), (3.5b) to the left-hand side of Eq.(3.24). Using Stokes' theorem (1.35) reverse-ly, we obtain the right-hand side of transformation (3.24).

3.15. The Referential Operators of Reynolds, Helmholtz, and Thomson

It is possible to denote concisely the two- and three-term expressions on the right-hand sides of transformations given in Secs. 3.12 to 3.14 with the help of the referential operators named after Reynolds, Helmholtz, and Thomson. They are functions of referential velocity and of the fields subjected to operation. Here are the names, symbols and optional forms of the referential operators:

$$(3.25a) \quad \left. \begin{array}{l} \text{the Reynoldsian} \\ \text{(of a scalar)} \end{array} \right\} \quad \overrightarrow{\text{reyn}}_{\bar{w}} \alpha = \frac{\partial \alpha}{\partial t} + \text{div}(\bar{w} \alpha) = \frac{d\alpha}{dt} \bar{w} + \alpha \text{div} \bar{w}$$

$$(3.25b) \quad \left. \begin{array}{l} \text{the Reynoldsian} \\ \text{(of a vector)} \end{array} \right\} \quad \overrightarrow{\text{reyn}}_{\bar{w}} \bar{A} = \frac{\partial \bar{A}}{\partial t} + \overline{\text{div}} \bar{w} \bar{A} = \frac{d\bar{A}}{dt} \bar{w} + \bar{A} \text{div} \bar{w}$$

$$(3.25c) \quad \text{the Helmholtzian} \quad \overrightarrow{\text{helm}}_{\bar{w}} \bar{A} = \frac{\partial \bar{A}}{\partial t} + \bar{w} \text{div} \bar{A} - \overline{\text{rot}} \bar{w} \times \bar{A} = \\ = \frac{\partial \bar{A}}{\partial t} + \overline{\text{div}} \bar{w} \bar{A} - \overline{\bar{A} \cdot \overline{\text{grad}} \bar{w}} = \frac{d\bar{A}}{dt} \bar{w} + \bar{A} \text{div} \bar{w} - \overline{\bar{A} \cdot \overline{\text{grad}} \bar{w}}$$

$$(3.25d) \quad \text{the Thomsonian} \quad \overrightarrow{\text{thom}}_{\bar{w}} \bar{A} = \frac{\partial \bar{A}}{\partial t} + \overline{\text{grad}}(\bar{w} \bar{A}) - \overline{\bar{w} \times \overline{\text{rot}} \bar{A}} = \\ = \frac{d\bar{A}}{dt} \bar{w} + \overline{\text{grad}}(\bar{w} \bar{A}) - \overline{\bar{w} \cdot \overline{\text{agrad}} \bar{A}} = \frac{d\bar{A}}{dt} \bar{w} + \bar{A} \cdot \overline{\text{agrad}} \bar{w}$$

At $\bar{w} = 0$, the referential operators (now deprived of the lower arrow symbol) are simply equal to the local derivatives of fields:

$$(3.26) \quad \text{reyn } \alpha = \partial \alpha / \partial t, \quad \text{reyn } \bar{A} = \text{helm } \bar{A} = \text{thom } \bar{A} = \partial \bar{A} / \partial t$$

The referential operators satisfy the following relations:

$$(3.27a) \quad \overrightarrow{\text{reyn}} \bar{A} - \overrightarrow{\text{helm}} \bar{A} = \overline{A' \text{grad } \bar{w}}$$

$$(3.27b) \quad \overrightarrow{\text{reyn}} \bar{A} - \overrightarrow{\text{thom}} \bar{A} = \bar{A} \text{ div } \bar{w} - \overline{A' \text{agrad } \bar{w}}$$

$$(3.27c) \quad \overrightarrow{\text{thom}} \bar{A} + \overrightarrow{\text{helm}} \bar{A} - \overrightarrow{\text{reyn}} \bar{A} = \frac{d\bar{A}}{dt} \bar{w} + \overline{A \times \text{rot } \bar{w}}$$

$$(3.28a) \quad \overrightarrow{\text{reyn}} \text{div } \bar{A} = \text{div } \overrightarrow{\text{helm}} \bar{A} = \text{div} \left(\frac{\partial \bar{A}}{\partial t} + \bar{w} \text{div } \bar{A} \right)$$

$$(3.28b) \quad \overrightarrow{\text{helm}} \text{rot } \bar{A} = \text{rot } \overrightarrow{\text{thom}} \bar{A} = \text{rot} \left(\frac{\partial \bar{A}}{\partial t} - \bar{w} \times \text{rot } \bar{A} \right)$$

Substituting the referential velocity \bar{w} for \bar{A} in Eq.(3.25c) and taking the rule (1.15b) into account, we obtain

$$(3.29) \quad \overrightarrow{\text{helm}} \bar{w} = \frac{\partial \bar{w}}{\partial t} + \bar{w} \text{div } \bar{w}$$

Exercise 3.8. Prove the equivalence of the alternative forms of the Reynoldsians (3.25a,b).

Instruction. Use the rules (3.3a,b), (1.30a), and (1.31b).

Exercise 3.9. Prove the equivalence of the first and second form of the Helmholtzian (3.25c).

Instruction. Use the rules (1.32a) and (1.31b).

Exercise 3.10. Prove the equivalence of all three forms of the Thomsonian (3.25d).

Instruction. Use the rules (1.33), (3.3b), and (1.28a).

Exercise 3.11. Prove Eq.(3.27c).

Instruction. Use the rule (1.33).

Exercise 3.12. Prove the transformation (3.28a).

Instructions. Substitute $\alpha = \text{div } \bar{A}$ into Eq.(3.25a), consider Eq.(3.5a), and use the rule (1.30d) in Eq.(3.25c).

Exercise 3.13. Prove the transformation (3.28b).

Instructions. Replace \bar{A} by $\overrightarrow{\text{rot}} \bar{A}$ in Eq.(3.25c), consider Eqs. (1.30d), (3.5b), and use the rule (1.32b) in Eq.(3.25d).

3.16. The Leibniz Transformation Group

As a matter of fact, the transformations (3.17a,b), (3.21), and (3.23) originate from Leibniz's formula for the derivative of an integral within variable limits. They form the Leibniz's group of transformations, which, as denoted concisely with the help of referential operators (3.25), is as follows:

$$\begin{array}{ll}
 (3.30a) & \frac{d}{dt} \int_{\substack{V \\ \rightarrow \bar{w}}} \alpha \, d^3V = \int_{\substack{V \\ \rightarrow \bar{w}}} \text{reyn}_{\rightarrow \bar{w}} \alpha \, d^3V \\
 (3.30b) & \frac{d}{dt} \int_{\substack{V \\ \rightarrow \bar{w}}} \bar{A} \, d^3V = \int_{\substack{V \\ \rightarrow \bar{w}}} \overrightarrow{\text{reyn}} \bar{A} \, d^3V \\
 (3.30c) & \frac{d}{dt} \int_{\substack{S \\ \rightarrow \bar{w}}} (\bar{A} \, d^2\bar{s}) = \int_{\substack{S \\ \rightarrow \bar{w}}} (\overrightarrow{\text{helm}} \bar{A} \, d^2\bar{s}) \\
 (3.30d) & \frac{d}{dt} \int_{\substack{l \\ \rightarrow \bar{w}}} (\bar{A} \, d\bar{l}) = \int_{\substack{l \\ \rightarrow \bar{w}}} (\overrightarrow{\text{thom}} \bar{A} \, d\bar{l})
 \end{array}$$

In the cases of shell $\substack{S \\ \rightarrow \bar{w}}$ surrounding a region $\substack{V \\ \rightarrow \bar{w}}$, and of contour $\substack{l \\ \rightarrow \bar{w}}$ bordering an open surface $\substack{S \\ \rightarrow \bar{w}}$, the transformations (3.30c,d) may take alternative forms (see Exs. 3.14, 15):

$$\begin{array}{ll}
 (3.31a) & \frac{d}{dt} \oint_{\substack{S \\ \rightarrow \bar{w}}} (\bar{A} \, d^2\bar{s}) = \int_{\substack{V \\ \rightarrow \bar{w}}} \text{div}_{\rightarrow \bar{w}} \overrightarrow{\text{helm}} \bar{A} \, d^3V = \int_{\substack{V \\ \rightarrow \bar{w}}} \text{reyn}_{\rightarrow \bar{w}} \text{div} \bar{A} \, d^3V \\
 (3.31b) & \frac{d}{dt} \oint_{\substack{l \\ \rightarrow \bar{w}}} (\bar{A} \, d\bar{l}) = \int_{\substack{S \\ \rightarrow \bar{w}}} (\overrightarrow{\text{rot}} \overrightarrow{\text{thom}} \bar{A} \, d^2\bar{s}) = \int_{\substack{S \\ \rightarrow \bar{w}}} (\overrightarrow{\text{helm}} \overrightarrow{\text{rot}} \bar{A} \, d^2\bar{s})
 \end{array}$$

At $\bar{w} = 0$, the referential geometric objects become the immobile integration regions V , s , l , and the referential derivatives and operators take shape of partial time-derivatives (Eqs.3.1e, 3.26), which can be transferred across the integration sign directly:

$$(3.32) \quad \frac{\partial}{\partial t} \int_{V,s,l} \dots d\dots = \int_{V,s,l} \frac{\partial \dots}{\partial t} d\dots$$

If at any point of the region the function is equal to zero, then its integral over the same region must be zero. If the integral mentioned is a time-derivative of another integral, then the latter integral must keep constant value in time. As applied to transformations (3.30), this leads to the following theorems:

If at any point of the region of integration

$$(3.33a) \quad \overrightarrow{\text{reyn}}_{\vec{w}} \alpha = 0, \text{ then } \int\limits_{\vec{V} \rightarrow \vec{w}} \alpha d^3V = \text{const}\langle t \rangle$$

$$(3.33b) \quad \overrightarrow{\text{reyn}}_{\vec{w}} \bar{A} = 0, \text{ then } \int\limits_{\vec{V} \rightarrow \vec{w}} \bar{A} d^3V = \text{const}\langle t \rangle$$

$$(3.33c) \quad \overrightarrow{\text{helm}}_{\vec{w}} \bar{A} = 0, \text{ then } \int\limits_{\vec{S} \rightarrow \vec{w}} (\bar{A} d^2\vec{s}) = \text{const}\langle t \rangle$$

$$(3.33d) \quad \overrightarrow{\text{thom}}_{\vec{w}} \bar{A} = 0, \text{ then } \int\limits_{\vec{l} \rightarrow \vec{w}} (\bar{A} d\vec{l}) = \text{const}\langle t \rangle$$

Reciprocal theorems are not, in general, valid. A time-constant value of an integral does not mean that the appropriate referential operator is equal to zero over all the region of integration.

Exercise 3.14. Derive the transformation (3.31a).

Instruction. Apply the GGO theorem (1.34a), transformation (3.30a) and Eq.(3.28a) to the left-hand side of Eq.(3.31a).

Exercise 3.15. Derive the transformation (3.31b).

Instruction. Apply Stokes' theorem (1.35), transformation (3.30c) and Eq.(3.28b) to the left-hand side of Eq.(3.31b).

3.17. Some Applications of the Leibniz Transformation Group

Because of the practical needs of the fluid mechanics, the Leibniz transformation group has been formulated for the first time as "Reynold's transport theorem" (Eqs.3.17 at $\vec{w} = \vec{u}$), "Helmholtz's vorticity theorem" (Ex. 3.17), and "Thomson's circulation theorem" (Ex. 3.21). Since the referential velocity was still unknown in the 19th century, those theorems concerned particular cases only. To-day

in their generalized forms, the transformations (3.30) have versatile applications to the balance theory (Chapter 4) and continuum mechanics, as well as to the nonrelativistic electrodynamics (Exs. 3.18, 19, 20, 23, 25).

Exercise 3.16. Prove that for a referential region the dilatation rate (or the rate of relative change of volume) is

$$(3.34) \quad \frac{1}{d^3V} \frac{d(d^3V)}{dt} \rightarrow \bar{w} = \text{div } \bar{w}$$

Instruction. Substitute $\alpha = 1$ into the transformation (3.17a), and confine the integration region to elementary cell of volume d^3V .

Exercise 3.17. Prove Helmholtz's theorem stating that in an incompressible fluid of specific volume $v = 1/\rho = \text{const}$ and shear viscosity coefficient $\eta_s = \text{const}$, moving with velocity \bar{u} , the vorticity $\bar{\omega} = \frac{1}{2} \text{rot } \bar{u}$ satisfies the relation

$$(3.35) \quad \frac{d\bar{\omega}}{dt} \rightarrow \bar{u} = \overline{\bar{\omega} \cdot \bar{g} \bar{r} \bar{a} \bar{d}} \bar{u} + \frac{1}{2} v \eta_s \overline{\text{div } \bar{g} \bar{r} \bar{a} \bar{d}} \bar{\omega}$$

Instructions. The described fluid complies with the Navier-Stokes equation (5.33a) (given in a further chapter), which, after substituting Eq.(5.23b) and transferring $v = \text{const}$ behind the gradient sign, may be presented as follows:

$$(3.36) \quad \frac{d\bar{u}}{dt} \rightarrow \bar{u} = \frac{1}{2} v \eta_s \overline{\text{div } \bar{g} \bar{r} \bar{a} \bar{d}} \bar{u} - \overline{\text{grad}(\Gamma + pv)}$$

According to the rules (3.3b), (1.28b), (1.33), (3.5b), and (1.32b), we have

$$\frac{d\bar{u}}{dt} \rightarrow \bar{u} = \frac{\partial \bar{u}}{\partial t} + \overline{\text{grad}(\frac{\bar{u}^2}{2})} - 2 \overline{\bar{u} \times \bar{\omega}}, \quad \frac{1}{2} \overline{\text{rot}} \frac{d\bar{u}}{dt} \rightarrow \bar{u} = \frac{\partial \bar{\omega}}{\partial t} - \overline{\text{rot } \bar{u} \times \bar{\omega}}$$

The right-hand side of the latter equation is the first form of the substantial Helmholtzian (3.25c), because of $\text{div } \bar{\omega} = 0$ (see Eq.1.30d). Since the substantial region $V_{\bar{u}}$ filled with incompressible fluid satisfies the requirement (3.12a) at $\bar{w} = \bar{u}$, the Helmholtzian can be expressed by the third form of Eq.(3.25c) as well.

Finally, according to rules (1.32b,d), one-half of the curl of the right-hand side of Eq.(3.36) gives $\frac{1}{2} \nabla \eta_s \overline{\text{div}} \overline{\overline{\text{grad}}} \overline{\omega}$.

Exercise 3.18. Prove that in a conducting fluid of resistivity $\eta = \text{const}$ and permeability $\mu = \text{const}$, moving with velocity \vec{u} , the magnetic induction \vec{B} satisfies the relation

$$(3.37) \quad \overline{\text{helm}}_{\vec{u}} \vec{B} = \frac{c^2 \eta}{\mu} \overline{\text{div}} \overline{\overline{\text{grad}}} \vec{B}$$

(the constant c is equal to velocity of light in vacuum).

Instructions. We start from the three Maxwell's equations

$$(3.38a) \quad \overline{\text{rot}} \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \quad (3.38b) \quad \overline{\text{rot}} \vec{H} = \frac{1}{c} \vec{I}, \quad (3.38c) \quad \text{div} \vec{B} = 0$$

and two phenomenological relations

$$(3.39a) \quad \vec{B} = \mu \vec{H}, \quad (3.39b) \quad \eta \vec{I} = \vec{E} + \frac{1}{c} \overline{\vec{u} \times \vec{B}}$$

where \vec{E} and \vec{H} denote the vectors of electric and magnetic field, respectively, and \vec{I} stands for the current density. While Eqs. (3.38a,c) are universally valid, Eq.(3.38b) keeps sufficient accuracy for a conducting medium, being a simplified form of Eq.(4.35b) written in a further chapter.

The magnetohydrodynamic fluids (plasma, liquid metal) comply with the approximate relations (3.39a,b). Eq.(3.39b) is a specific form of Ohm's law (see Eqs.4.45 and 4.33a,b, given in a further chapter) for electrically neutral ($\rho_e = 0$) and homogenous ($\overline{\vec{K}}_{\vec{u}} = 0$) fluid.

We compute the curl of Eq.(3.39b), substitute Eqs.(3.38a,b,c) and (3.39a), then use the rule (1.32c), and obtain

$$(3.40) \quad \frac{\partial \vec{B}}{\partial t} - \overline{\text{rot}} \overline{\vec{u} \times \vec{B}} = \frac{c^2 \eta}{\mu} \overline{\text{div}} \overline{\overline{\text{grad}}} \vec{B}$$

The left-hand side of the result is the first form of the substantial Helmholtzian (3.25c), because of the condition (3.38c).

Exercise 3.19. Transform the relation (3.37) into

$$(3.41a) \quad \frac{d\vec{B}}{dt} \rightarrow \vec{u} = \overline{\vec{B} \cdot \vec{g} \vec{r} \vec{a} \vec{d}} \vec{u} - \vec{B} \operatorname{div} \vec{u} + \frac{c^2 \mu}{\mu} \operatorname{div} \vec{g} \vec{r} \vec{a} \vec{d} \vec{B}$$

$$(3.41b) \quad \frac{d(v\vec{B})}{dt} \rightarrow \vec{u} = \overline{(v\vec{B}) \cdot \vec{g} \vec{r} \vec{a} \vec{d}} \vec{u} + \frac{c^2 \mu v}{\mu} \operatorname{div} \vec{g} \vec{r} \vec{a} \vec{d} \vec{B}$$

where $v = 1/\rho$ is the specific volume of fluid.

Instruction re Eq.(3.41b). Use Eq.(5.4c) given in a further chapter.

Exercise 3.20. Deduce the following nonrelativistic relation between electric field vectors $\vec{E}_{\rightarrow \vec{w}}$ and \vec{E} , assigned to the referential and immobile point, respectively (at the instant of covering each other):

$$(3.42) \quad \vec{E}_{\rightarrow \vec{w}} = \vec{E} + \frac{1}{c} \overline{\vec{w} \times \vec{B}}$$

making no use of the relativistic Lorentz transformation.

Instructions. The starting point is Faraday's integral law of induction presented in terms of referentiality

$$(3.43) \quad \underbrace{\oint_{\vec{l}_{\rightarrow \vec{w}}} (\vec{E}_{\rightarrow \vec{w}} \cdot d\vec{l})}_{\text{electromotive force along contour } \vec{l}_{\rightarrow \vec{w}}} = - \frac{1}{c} \frac{d}{dt} \underbrace{\iint_{\vec{S}_{\rightarrow \vec{w}}} (\vec{B} \cdot d^2\vec{s})}_{\text{magnetic induction flux through open surface } \vec{S}_{\rightarrow \vec{w}} \text{ bordered by contour } \vec{l}_{\rightarrow \vec{w}}}$$

To the left-hand side we apply Stokes' theorem (1.35), and in the right-hand side we use the Leibniz-Helmholtz transformation (3.21), with the substitution of universally valid Maxwell's equations (3.38a,c). Transferring all the terms to one side, we obtain a surface integral which is equal to zero for arbitrarily small surface, because of the assumed continuity of fields.

Hence the integrated function must vanish always and everywhere: $\operatorname{rot}(\vec{E}_{\rightarrow \vec{w}} - \vec{E} - \frac{1}{c} \overline{\vec{w} \times \vec{B}}) = 0$. According to the rule (1.32b), the solution of this equation assumes the shape of gradient of any scalar field: $\vec{E}_{\rightarrow \vec{w}} - \vec{E} - \frac{1}{c} \overline{\vec{w} \times \vec{B}} = \vec{g} \vec{r} \vec{a} \vec{d} \alpha$. We cannot, however, accept the infinite number of solutions, because, from the physical viewpoint

all the quantities on the left-hand side are univocally definite and experimentally measurable. The only physical solution results from substitution $\alpha = 0$. The final result (3.42) keeps sufficient accuracy in the nonrelativistic conditions ($|\bar{w}| \ll c$).

Additional exercise. Show how to obtain Maxwell's equation (3.38a) from Faraday's law (3.43) at $\bar{w} = 0$.

Exercise 3.21. Prove Thomson's theorem stating that the circulation integral of velocity of the incompressible and inviscid fluid along a substantial contour remains constant in time:

$$(3.44) \quad \oint_{\vec{l} \rightarrow \vec{a}} (\bar{u} \, d\bar{l}) = \text{const} \langle t \rangle$$

Instructions. If we apply Stokes' theorem (1.35) to the left-hand side of Eq.(3.44) and take into account the theorem (3.33c) and Eq.(3.28b), then nothing else is required than to prove that $\overrightarrow{\text{helm}} \overrightarrow{\text{rot}} \bar{u} = \overrightarrow{\text{rot}} \overrightarrow{\text{thom}} \bar{u} = 0$, or, according to the third form of Thomsonian (3.25d), that $\overrightarrow{\text{rot}} \left(\frac{d\bar{u}}{dt} \right) + \bar{u}' \overrightarrow{\text{agr}} \bar{u} = 0$. This can be achieved by applying Eq.(3.36) at $\eta_s = 0$ (inviscid fluid) and using the rules (1.28b) and (1.32b).

3.18. Convection and Diffusion of the Vector Field Lines

The geometric image of a vector field \bar{A} consists of set of lines, to which the vector \bar{A} is tangent at every point. A small arc $\delta \bar{l}$ of the field line and an identically situated field vector \bar{A} are parallel:

$$(3.45) \quad \delta \bar{l} = \delta \alpha \bar{A}$$

$\delta \alpha$ being a scalar proportionality factor.

In some circumstances the time-space changes of the vector field \bar{A} may be interpreted as convection of all its lines. They are being carried away with the convective region $\vec{V}_{\bar{w}_c}$ which moves in accordance with some particular velocity field \bar{w}_c . In such case, as it

results from the substitution of Eq.(3.45) into the relation (3.10a) at $\vec{w} = \vec{w}_c$, we have

$$(3.46) \quad \frac{d\delta\vec{l}}{dt} \rightarrow \vec{w}_c = \overline{\delta\vec{l} \cdot \vec{g} \vec{r} \vec{a} \vec{d}} \vec{w}_c = \delta\alpha \frac{d\vec{A}}{dt} \rightarrow \vec{w}_c + \vec{A} \frac{d\delta\alpha}{dt} \rightarrow \vec{w}_c$$

The velocity field \vec{w}_c must not be arbitrary and sometimes cannot be uniquely defined. We shall confine our reasoning to a case ruled by the following theorem:

$$(3.47) \quad \text{If a vector field } \vec{A} \text{ satisfies the relation } \frac{d\vec{A}}{dt} \rightarrow \vec{w}_c = \overline{\vec{A} \cdot \vec{g} \vec{r} \vec{a} \vec{d}} \vec{w}_c \text{ at some particular velocity field } \vec{w}_c, \text{ then the convection of the field } \vec{A} \text{ occurs with velocity } \vec{w}_c, \text{ and the factor } \delta\alpha = \text{const at each point of the convective region } V_{\vec{w}_c}.$$

Leaving the rigorous proof out, we shall merely mention that if the same type of equation describes the changes of vectors $\delta\vec{l}$ and \vec{A} (defined by Eqs.3.46 and 3.47), then at each convective point the vector \vec{A} is not only parallel to $\delta\vec{l}$, but also the ratio of magnitudes of both vectors remains constant. If in its motion a small segment $\delta\vec{l}$ of the field line grows longer, then the magnitude of \vec{A} increases proportionally. Indeed, if we substitute Eqs.(3.47) and (3.45) into Eq.(3.46), then obviously $\frac{d\delta\alpha}{dt} \rightarrow \vec{w}_c = 0$.

If a vector field \vec{A} satisfies the diffusion-type equation

$$(3.48) \quad \frac{\partial \vec{A}}{\partial t} = \beta \overline{\text{div } \vec{g} \vec{r} \vec{a} \vec{d}} \vec{A}$$

where β is the diffusion coefficient, then the changes of the field lines are no more interpreted as a motion but as a diffusive "flow-away" (i.e. spatial equalizing of the density of field lines with the time lapsing).

If a vector field \vec{A} satisfies the following type of equation

$$(3.49) \quad \frac{d\vec{A}}{dt} \rightarrow \vec{w}_c = \underbrace{\overline{\vec{A} \cdot \vec{g} \vec{r} \vec{a} \vec{d}} \vec{w}_c}_{\text{convection of the vector field lines}} + \underbrace{\beta \text{div } \vec{g} \vec{r} \vec{a} \vec{d} \vec{A}}_{\text{diffusion}}$$

then the changes of the field lines may be interpreted as the simultaneous convection and diffusion. At $\beta = 0$, the relation (3.49) takes form of equation of the pure convection (3.47), and at $\bar{w}_c = 0$, of equation of the pure diffusion (3.48).

In particular circumstances the fields of the vorticity $\bar{\omega}$ and magnetic induction \bar{B} are subjected to pure convection in a fluid streaming with velocity \bar{u} (Exs. 3.22, 23). The lines of those fields keep "stuck" to the substance and are being deformed in the same way as the fluid. A substantial point once situated on a particular line of field $\bar{\omega}$ or \bar{B} will remain there forever.

Exercise 3.22. Prove that the vorticity field $\bar{\omega} = \frac{1}{2} \text{rot } \bar{u}$ of incompressible and inviscid fluid is subjected to pure convection with the velocity of substance \bar{u} .

Instruction. Use Eq.(3.35) at $\gamma_s = 0$ and apply the theorem (3.47)

Exercise 3.23. Prove that in a compressible but perfectly conducting fluid, the field $v\bar{B} = \bar{B}/\varrho$ ($v = 1/\varrho$ denotes the specific volume, and \bar{B} the magnetic induction) is subjected to pure convection with the velocity of substance \bar{u} .

Instruction. Use Eq.(3.41b) at infinite conductivity $1/\mu$, and apply the theorem (3.47).

Exercise 3.24. Prove in two different ways that the flux of vorticity $\bar{\omega}$ through a substantial surface $\vec{s}_{\bar{u}}$ remains constant in time

$$(3.50) \quad \iint_{\vec{s}_{\bar{u}}} (\bar{\omega} d^2\vec{s}) = \text{const}\langle t \rangle \quad \text{for incompressible and inviscid fluid}$$

Instructions. Compute $\overline{\text{helm}}_{\vec{s}_{\bar{u}}} \bar{\omega}$ from the third form of Eq.(3.25c), consider the incompressibility condition (3.12a) at $\bar{w} = \bar{u}$, and Eq. (3.35) at $\gamma_s = 0$, then use the theorem (3.33c). The second method lies in substituting $\bar{\omega} = \frac{1}{2} \text{rot } \bar{u}$ into the left-hand side of Eq. (3.50), and applying the theorems of Stokes (1.35) and Thomson (3.4.

Exercise 3.25. Prove that the electromotive force along a substantial contour $\vec{l}_{\bar{u}}$ vanishes

$$(3.51) \quad \oint_{\vec{u}} (\vec{E} \cdot d\vec{l}) = 0 \quad \text{in perfectly conducting fluid.}$$

Instructions. Compute $\overline{\text{helm}}_{\vec{u}} \vec{B}$ from the third form of Eq.(3.25c), take into account Eq.(3.41a) and the condition of perfect conductivity ($\mathcal{H} = 0$), then apply the theorem (3.33c) and Faraday's law of induction (3.43).

4. The Balances of Extensive Quantities

4.1. The Extensive Quantity, its Storage, Production and Transfer

The extensive quantity, or EQ, as abbreviated, is a geometric or physical quantity which can be stored in a region. The amount of stored EQ will be called the storage. At any arbitrary division of a region into sub-regions, the storage ascribed to the whole region is a sum of storages of all the sub-regions. Such a property is peculiar to all the scalar EQs, e.g. the volume, mass, electric charge, energy, or entropy, and to all the vector EQs, e.g. the momentum, or angular momentum.

The site of an EQ is the space, either filled with substance, or not. In the traditional thermodynamic nomenclature, the site of EQ is restricted to the mass only. This particular kind of EQ, according to our classification, will be called the substantial quantity (see Chapter 5).

The EQ is an integral quantity ascribed to a system formed of some spatial region of finite dimensions. The distribution of a continuous EQ in space is described by the field of the storage density of EQ.

A contrary to the EQ is an intensive quantity (e.g. the temperature, or pressure), which cannot form any storage in a region. Being assigned to a point in space, a field quantity is never an EQ. Thus, the storage density of an EQ ceases to be the extensive

quantity, and becomes an intensive quantity, though it sounds paradoxically.

Fig. 4.1 We distinguish two causes of change of the EQ stored in a system (Fig. 4.1).

The first one concerns the phenomena occurring solely within the system and giving occasion to creation or annihilation of EQ. Both processes considered together are called the production of EQ.

A particular EQ can be created or annihilated at the expense or for the benefit of another EQ coexisting within the same system. Such a process is called the conversion between two or among more EQs. The sum of productions of mutually convertible EQs, forming a closed set within the system, is always zero.

The second cause of change of the storage is the interaction between the system and its surroundings, and this occurs across the boundary of the system. In case of a scalar EQ, e.g. the mass or energy, such an interaction may be pictured as a flow, viz. an inflow into the system, or an outflow to the surroundings. In case of the momentum, which is a vector EQ, the interaction takes shape of the surface force. All the processes mentioned are covered by common name of transfer of EQ between the system and its surroundings.

In case of a positive-valued EQ, the creation and the inflow contribute to the increase of storage, while the annihilation and the outflow cause the decrease of storage, so the following sign convention will stand:

$$\begin{aligned} \text{creation} &> 0, & \text{annihilation} &< 0, \\ \text{inflow} &> 0, & \text{outflow} &< 0, \\ \text{production} &= \text{creation} + \text{annihilation} \geq 0, \\ \text{transfer} &= \text{inflow} + \text{outflow} \geq 0. \end{aligned}$$

The sign convention becomes useless in case of a vector EQ. Its production and transfer are no longer scalars but vectors, defined

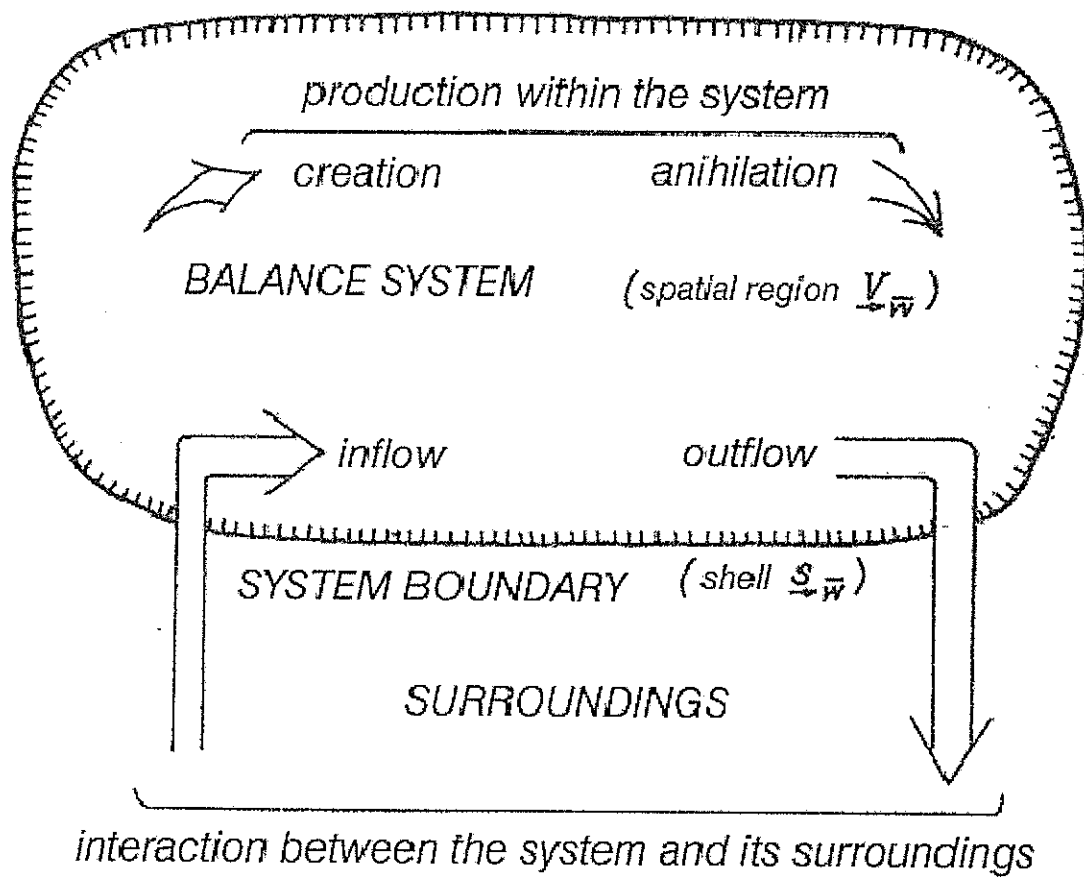


Fig. 4.1. Scheme of causes changing the storage of extensive quantity

by direction instead of sign. For example a body moving along the circular orbit experiences a "transverse" production of momentum.

4.2. The Balance Axiom and the Basic Balance

Since the causes of change of the storage have been classified, the next step is to formulate the following balance axiom for an EQ:

$$(4.1) \left\{ \begin{array}{c} \text{change of EQ stored} \\ \text{within system} \end{array} \right\} = \left\{ \begin{array}{c} \text{EQ produced} \\ \text{within system} \end{array} \right\} + \left\{ \begin{array}{c} \text{EQ transferred} \\ \text{between system and} \\ \text{surroundings through} \\ \text{system's boundary} \end{array} \right\}$$

Relating the given balance to the unit time, we obtain the rates of change of storage, production, and transfer, as particular terms. Being deliberately not too rigorous in the meaning of words for the benefit of conciseness, the rates of change of production and transfer we shall call simply the "production" and the "transfer", respectively, bearing in mind that they are already related to the unit time.

Denoting the storage, production, and transfer of a scalar EQ by $\mathcal{S}, \mathcal{P}, \mathcal{T}$, respectively, and the same notions for a vector EQ by $\vec{\mathcal{S}}, \vec{\mathcal{P}}, \vec{\mathcal{T}}$, respectively, we translate the verbal equation (4.1) into a balance carried out during time dt in an immobile system:

$$(4.2a) \quad d\mathcal{S} = \mathcal{P} dt + \mathcal{T} dt, \quad (4.2b) \quad d\vec{\mathcal{S}} = \vec{\mathcal{P}} dt + \vec{\mathcal{T}} dt$$

Considering a more general case of the mobile system, we meet a question whether the terms of balance are referential quantities, or not. The motion of boundary makes the storage of surroundings captured by the system, or vice versa. For that reason the transfer through the boundary of a referential system $(\vec{\mathcal{T}}_{\rightarrow}, \vec{\mathcal{T}}_{\leftarrow})$ is not the same as in case of the immobile system, at the instant when both systems cover each other.

In the nonrelativistic conditions the production and the storage are not influenced by the motion of system (they keep non-referential character). In a mobile system, however, the change of

storage during time dt becomes a referential increment $d\vec{S}_{\vec{w}}$, $d\vec{S}_{\vec{w}}$ of the non-referential quantity S , \bar{S} : since the boundary has been displaced by $\vec{w} dt$, the contribution of slice of that thickness must be taken into account.

Consequently, the balance for a referential system is

$$(4.3a) \quad d\vec{S}_{\vec{w}} = \vec{P} dt + \vec{T}_{\vec{w}} dt, \quad (4.3b) \quad d\vec{S}_{\vec{w}} = \bar{P} dt + \bar{T}_{\vec{w}} dt$$

From now on, we shall only deal with the balances related to the unit time. Dividing Eqs.(4.3) by dt , we obtain the basic balances:

$$(4.4a) \quad \frac{d\vec{S}_{\vec{w}}}{dt} = \vec{P} + \vec{T}_{\vec{w}} \quad \text{for scalar EQ}$$

$$(4.4b) \quad \underbrace{\frac{d\vec{S}_{\vec{w}}}{dt}}_{\substack{\text{rate of change} \\ \text{of storage}}} = \underbrace{\bar{P}}_{\text{production}} + \underbrace{\bar{T}_{\vec{w}}}_{\text{transfer}} \quad \text{for vector EQ}$$

The analysis of annual change of number of cars in some country is a demonstrative example of the balance (4.2a). The change results from the manufactured output (creation) and scrapping (annihilation) within the country, as well as from the foreign trade (transfer): the importation (inflow) and the exportation (outflow).

The balance (4.3a) may be illustrated by the change of number of the vehicles in the circumstances of war, when some of the country's frontiers are mobile fronts (boundaries of referential region). In such circumstances, the transfer should contain the capture effect, resulting from the equipment taken from the enemy in attack, and abandoned for enemy's benefit in retreat.

4.3. The Continuous Extensive Quantity

From now on, our considerations will be restricted to the continuous EQs. Their densities of storage, production, and transfer are assumed to be time-space-continuous and differentiable fields. However, we must keep the phenomenological character of continuity and differentiability in mind, because the densities themselves

are the apparent limits (limfs, see Sec. 2.1) for very small but not infinitesimal regions.

The three-dimensional region is the site of storage of EQ, as well as the place where the production of EQ occurs. Since the volume is a scalar measure for such a region, the densities of storage and production are defined as follows:

Category of the (volume-) density	Definition	Denotation for	
		scalar EQ	vector EQ
(4.5) $\left. \begin{array}{l} \text{storage} \\ \text{density of EQ} \end{array} \right\}$	$= \limf \frac{\text{stored EQ}}{\text{volume}}$	ψ	\bar{P}
(4.6) $\left. \begin{array}{l} \text{production} \\ \text{density of EQ} \end{array} \right\}$	$= \limf \frac{\text{produced EQ}}{\text{time} \cdot \text{volume}}$	Ψ	\bar{R}

The "density" deprived of any adjective will be meant as the "volume-density of storage". In cases of the substantial and componential quantities, we shall also use the mass-densities and the partial densities.

Since the storage and production are non-referential quantities, their densities do not depend on the motion of the balance system. Consequently, the storage and production of the scalar and vector EQ within a referential region are expressed as follows:

$$(4.7a) \quad \mathcal{S} = \iiint_{\substack{V \\ \rightarrow \bar{w}}} \psi \, d^3v, \quad (4.7b) \quad \bar{\mathcal{S}} = \iiint_{\substack{V \\ \rightarrow \bar{w}}} \bar{P} \, d^3v$$

$$(4.8a) \quad \mathcal{P} = \iiint_{\substack{V \\ \rightarrow \bar{w}}} \Psi \, d^3v, \quad (4.8b) \quad \bar{\mathcal{P}} = \iiint_{\substack{V \\ \rightarrow \bar{w}}} \bar{R} \, d^3v$$

4.4. The Transport of an Extensive Quantity

In the balance written for a system, we consider the transfer through the boundary surface only. The transport process of EQ occurs, however, in the whole three-dimensional interior of the system. It results from the assumed continuity of EQ that at any

imaginary way of division of the balance region into even the smallest sub-regions, there exists a possibility of transfer across arbitrarily situated interfaces.

In the microphysical reality, the process of transport lies in the carrying of discrete storages of EQ by the grains of substance and photons, which move in various directions through very small imaginary spatial region. All the possible directions of velocities of EQ's carriers can be allocated to narrow cones, having a common vertex and producing the total solid angle 4π altogether. This way, the real transport of EQ is nothing else but a directional transmission.

Consider a flow of EQ across a very small surface element (compare Eq.1.23b)

$$(4.9a) \quad \delta^2 \bar{s}_{\Omega} = \bar{I}_{\Omega} \delta^2 s_{\Omega}$$

and within an infinite set of identical narrow cones with vertexes covering all the points of $\delta^2 \bar{s}_{\Omega}$. The cones are unidirected, each of them being described by the vectorial solid angle

$$(4.9b) \quad \delta^2 \bar{\Omega} = \bar{I}_{\Omega} \delta^2 \Omega$$

Fig. 4.2 Both $\delta^2 \bar{s}_{\Omega}$ and $\delta^2 \bar{\Omega}$ have the same unit vector \bar{I}_{Ω} , thus the axis of the solid angle is normal to the surface element (Fig. 4.2a). Since the surface element is imaginary, it does not hinder the transport of EQ in any direction. This time, however, we take into account only those velocity directions of EQ's carriers which are not much deviated with respect to the unit vector \bar{I}_{Ω} .

In order to examine the directional characteristic of transport, we swivel the cone set with the purpose to cover all possible directions of \bar{I}_{Ω} . While measuring the amount of transmitted EQ, we keep the cone axes perpendicular to the surface element (Fig.4.2b). Also the magnitudes of vectors $\delta^2 \bar{\Omega}$ (i.e. the scalar solid angle $\delta^2 \Omega$) and $\delta^2 \bar{s}_{\Omega}$ (i.e. the scalar area $\delta^2 s_{\Omega}$), as well as the time

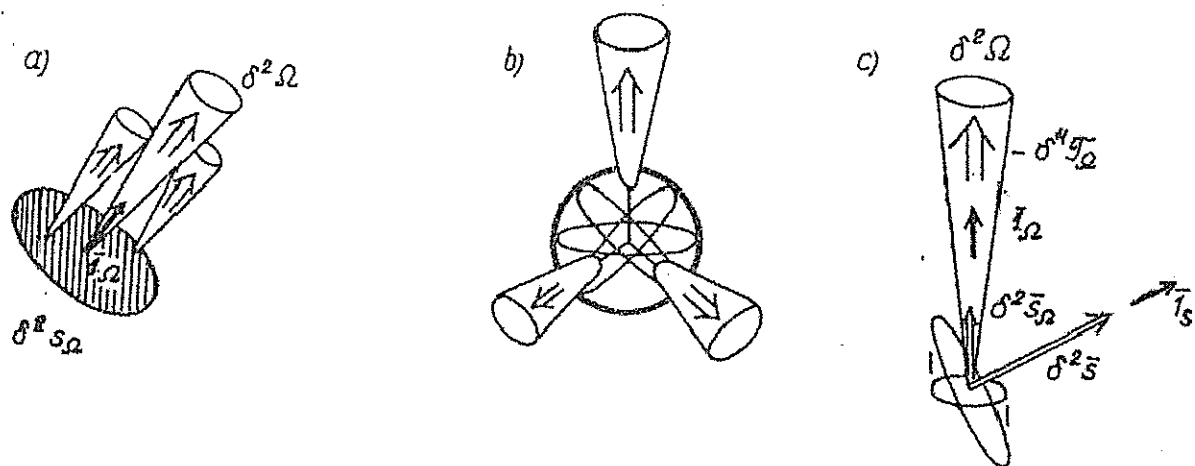


Fig. 4.2. Directional transmission of extensive quantity (flow directions of EQ are marked by broad arrows)

interval δt , are kept constant.

Within the apparent limit, the amount of EQ transmitted along the cones $\delta^2\Omega$ and across the surface element $\delta^2\bar{s}_\Omega$ during time δt remains proportional to three scalars: δt , δ^2s_Ω , and $\delta^2\Omega$. Consequently, we define the next category of density, which is:

$$(4.10) \quad \left. \begin{array}{l} \text{transmission} \\ \text{density of EQ} \end{array} \right\} = \limf \frac{\text{transmitted EQ}}{\text{time} \cdot \text{area} \cdot \text{solid angle}}$$

Denotation for $\begin{cases} \text{scalar EQ: } H_\Omega \\ \text{vector EQ: } \bar{T}_\Omega \end{cases}$

The density of transmission is a function of three variables: the unit vector \bar{T}_Ω of direction in space, the position \bar{r} in space, and the time t .

4.5. The Transfer-Flux Density

Since the directional transmission is something alike the centrifugal flow of EQ out of a very small sphere (Fig. 4.2b), it contributes to the negative transfer of EQ. According to definition (4.10), the EQ flown out of the sphere through its cross-section surface element $\delta^2\bar{s}_\Omega$ within the cone $\delta^2\Omega$ per unit time is:

$$(4.11a) - \delta^4\bar{T}_\Omega = H_\Omega \delta^2s_\Omega \delta^2\Omega, \quad (4.11b) - \delta^4\bar{T}_\Omega = \bar{T}_\Omega \delta^2s_\Omega \delta^2\Omega$$

Now we consider the flow of EQ through another very small imaginary surface element

$$(4.12a) \quad \delta^2\bar{s} = \bar{T}_s \delta^2s$$

of constant area δ^2s and constant direction \bar{T}_s . The centres of surface elements, both the fixed one $\delta^2\bar{s}$ and the swivelling one $\delta^2\bar{s}_\Omega$, are positioned identically, whereas the projection of $\delta^2\bar{s}$ on the plane normal to \bar{T}_Ω covers $\delta^2\bar{s}_\Omega$ precisely at any direction of \bar{T}_Ω (Fig. 4.2c). This way the area of the swivelling element is proportional to the cosine of the angle between \bar{T}_s and \bar{T}_Ω , or to the scalar product of unit vectors ($\bar{T}_s \bar{T}_\Omega$):

$$(4.12b) \quad \delta^2 s_{\Omega} = \delta^2 s (\mathbf{I}_s \mathbf{I}_{\Omega}) = (\delta^2 \overline{s} \mathbf{I}_{\Omega})$$

Now we apply Eqs.(4.12b) and (4.9b) to Eqs.(4.11), reach the apparent limit ($\delta \rightarrow d$), and use the rule (1.20a) to obtain the transfer of EQ through fixed surface element $d^2 \overline{s}$ within the cone $d^2 \Omega$:

$$(4.13a) \quad - d^4 \mathbf{T}_{\Omega} = (d^2 \overline{s} d^2 \Omega) \mathbf{H}_{\Omega}$$

$$(4.13b) \quad - d^4 \overline{\mathbf{T}}_{\Omega} = (d^2 \overline{s} d^2 \Omega) \overline{\mathbf{T}}_{\Omega} = d^2 \overline{s} d^2 \Omega \cdot \overline{\mathbf{T}}_{\Omega}$$

The transfer through the same surface element bit within all the cones covering the total solid angle is a double integral with respect to $d^2 \Omega$; placing $d^2 \overline{s}$, as a constant vector, before the integration sign, we have:

$$(4.14a) \quad - d^2 \mathbf{T} = (d^2 \overline{s} \oint d^2 \Omega \mathbf{H}_{\Omega}), \quad (4.14b) \quad - d^2 \overline{\mathbf{T}} = d^2 \overline{s} \oint d^2 \Omega \cdot \overline{\mathbf{T}}_{\Omega}$$

The directional integration of the transmission density gives a finite quantity which characterizes the spatial net flux of EQ, all possible directions being taken into account. The results of integration are called the transfer-flux densities of scalar and vector EQ:

$$(4.15a) \quad \overline{\mathbf{H}} = \oint d^2 \Omega \mathbf{H}_{\Omega}, \quad (4.15b) \quad \overline{\mathbf{T}} = \oint d^2 \Omega \cdot \overline{\mathbf{T}}_{\Omega}$$

While the transmission density of EQ ($\mathbf{H}_{\Omega}, \mathbf{T}_{\Omega}$) is of the same tensor rank as the EQ itself, the transfer-flux density ($\overline{\mathbf{H}}, \overline{\mathbf{T}}$) has the rank higher by one.

The directional integration reveals the predomination of some cones of transmission over the others, and makes possible to disclose the direction of the resultant transport of EQ.

Being an integrated product of a vector and a scalar, the vector $\overline{\mathbf{H}}$ shows the direction of the resultant transport of scalar EQ outright. In case of the transport of mass, the vector $\overline{\mathbf{H}}$ is proportional to the barycentric velocity $\overline{\mathbf{u}}$. When $\overline{\mathbf{u}} = 0$, it does not mean that there is no transmission of mass at all but it proves

either the directional compensation of transmission (the contributions of the opposite cones cancel each other), or isotropy of transmission (the contributions of all the cones are identical), peculiar to the purely thermal motion of grains of substance. Similarly, in case of the isotropic radiation, the energy is transmitted evenly in all the directions, resulting in the transfer-flux density equal to zero.

Being an integrated dyadic product of two vectors, the tensor $\bar{\bar{T}}$ conjugates the direction of the vector \mathbf{EQ} itself with the direction of its resultant transport. It is peculiar to the momentum transport in gas at macroscopic rest that for any transmission cone the vectors $\bar{\mathbf{T}}_{\Omega}$ and $d^2\bar{\mathbf{T}}$ are directed identically (Fig. 4.2b), and the magnitude of $\bar{\mathbf{T}}_{\Omega}$ is isotropic. Owing to this, the transfer-flux density $\bar{\bar{T}}$ becomes a spherical tensor $\bar{\mathbf{I}}p$ (the scalar p stands for the pressure of gas), having no distinguished directions.

While the transfer-flux density describes a three-dimensional transport process independent of any geometric reference, the elementary transfer depends on the direction of $d^2\bar{\mathbf{s}}$, as it results from substitution of Eqs(4.15) into Eqs.(4.14):

$$(4.16a) - d^2\bar{\mathbf{T}} = (d^2\bar{\mathbf{s}} \bar{\mathbf{H}}), \quad (4.16b) - d^2\bar{\mathbf{T}} = \overline{d^2\bar{\mathbf{s}} \cdot \bar{\bar{T}}}$$

If the resultant transport of a scalar \mathbf{EQ} occurs tangentially to the surface element $(\bar{\mathbf{H}} \perp d^2\bar{\mathbf{s}})$, there is no transfer through $d^2\bar{\mathbf{s}}$ at all, despite of non-zero flux density $\bar{\mathbf{H}}$. In macroscopically resting gas, the momentum transfer vector is always normal to the surface element $d^2\bar{\mathbf{s}}$ and equal to $-p d^2\bar{\mathbf{s}}$, as it results from substitution of $\bar{\bar{T}} = \bar{\mathbf{I}}p$ into Eq.(4.16b) and the use of rule (1.18d).

4.6. The Capture Effect

The exposition given in the preceding section has been based on the assumption that the densities $\bar{\mathbf{H}}$ and $\bar{\bar{T}}$ are assigned to a resting point, and the transfer is occurring through an immobile surface

element $d^2\bar{s}$. At any motion of the surface, however, the transfer will be altered by an additional effect of capture.

Fig. 4.3

Consider two surface elements: the immobile one $d^2\bar{s}$ and the referential one $d^2\bar{s} \xrightarrow{\bar{w}}$. At the instant t they cover each other and have identical orientation: $d^2\bar{s} \xrightarrow{\bar{w}}(t) = d^2\bar{s}$. At a later instant $t+dt$ the referential element is occupying a different position $d^2\bar{s} \xrightarrow{\bar{w}}(t+dt)$ (Fig. 4.3). Along its referential displacement $\bar{w} dt$, the surface element $d^2\bar{s} \xrightarrow{\bar{w}}$ has swept the volume element $d^3V = (d^2\bar{s} \bar{w} dt)$ (see Eq. 1.23d), and captured the storage Ψd^3V or $\bar{P} d^3V$ of the scalar or vector EQ, respectively. The storage captured per unit time is called the elementary capture of scalar or vector EQ, and denoted by $d^2\bar{J}_c$ or $d^2\bar{J}_c$, respectively:

$$(4.17a) \quad d^2\bar{J}_c = (d^2\bar{s} \bar{w})\Psi = (d^2\bar{s} \bar{w})\bar{\Psi}$$

$$(4.17b) \quad d^2\bar{J}_c = (d^2\bar{s} \bar{w})\bar{P} = \overline{d^2\bar{s} \bar{w} \bar{P}}$$

(in case of the vector EQ the rule 1.20a has been used).

If the vectors \bar{w} and $d^2\bar{s}$ (the latter assumed to be outwardly bound with respect to the balance region) form an acute angle, then the capture (4.17a) effects the growth of the storage of scalar EQ, as well as it increases the transfer, being an additional inflow into the region.

Consequently, the transfer $d^2\bar{J} \xrightarrow{\bar{w}}$, $d^2\bar{J} \xrightarrow{\bar{w}}$ through the referential surface element $d^2\bar{s} \xrightarrow{\bar{w}}$ is the transfer across the immobile surface element (both elements covering each other at the instant given) plus the capture:

$$(4.18a) \quad d^2\bar{J} \xrightarrow{\bar{w}} = d^2\bar{J} + d^2\bar{J}_c$$

$$(4.18b) \quad \underbrace{d^2\bar{J} \xrightarrow{\bar{w}}}_{\substack{\text{transfer through} \\ \text{referential} \\ \text{surface element}}} = \underbrace{d^2\bar{J}}_{\substack{\text{transfer through} \\ \text{immobile} \\ \text{surface element}}} + \underbrace{d^2\bar{J}_c}_{\substack{\text{capture through} \\ \text{referential} \\ \text{surface element}}}$$

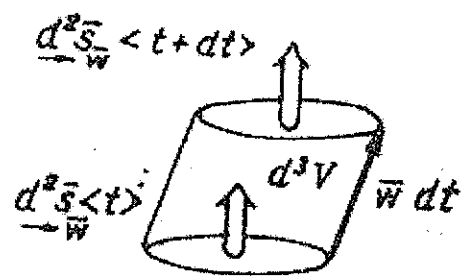


Fig. 4.3. Volume element swept by the referential surface element $d^2\vec{S}_w$

Substituting Eqs.(4.16) and (4.17) into Eqs.(4.18), we obtain the elementary transfer through a referential surface element (denoted simply as $d^2\vec{s}$, since it is frozen at the instant given)

$$(4.19a) \quad d\vec{J}_{\vec{w}} = - (d^2\vec{s} \cdot \vec{H}_{\vec{w}}), \quad (4.19b) \quad d\vec{J}_{\vec{w}} = - d^2\vec{s} \cdot \vec{T}_{\vec{w}},$$

expressed in terms of the referential transfer-flux density

$$(4.20a) \quad \vec{H}_{\vec{w}} = \vec{H} - \vec{w}\psi, \quad (4.20b) \quad \vec{T}_{\vec{w}} = \vec{T} - \vec{w}\vec{P}$$

where $\vec{w}\psi$ and $\vec{w}\vec{P}$ are the capture-flux densities.

The transfer-flux density at an immobile point is a particular case of the referential density, since

$$(4.21) \quad \vec{H}_{\vec{w}} = \vec{H}, \quad \vec{T}_{\vec{w}} = \vec{T} \quad \text{at } \vec{w} = 0$$

Fig. 4.4

The relation (4.18a) may be suggestively illustrated by the following example. Imagine an incompressible and inviscid fluid flowing along a pipe with the uniform velocity \vec{u} . The rate of flow, or the transfer of fluid's volume, is being measured at the control cross-section moving with velocity \vec{w} in the same direction as the fluid itself (Fig. 4.4). The rate of flow attains its largest (positive) value when the control cross-section is at rest. The increase in velocity \vec{w} causes subtraction of bigger and bigger capture effect from the rate of flow measured at rest. At $\vec{w} = \vec{u}$ the meter reading vanishes, and at $|\vec{w}| > |\vec{u}|$ the result of measurement becomes negative.

4.7. The Transfer as a Surface Interaction

The transfer of BQ through the shell of a referential balance region is represented by the integral of expression (4.19) over the closed surface, which, in turn, can be transformed into a volume integral, in accordance with the GGO theorem (1.34):

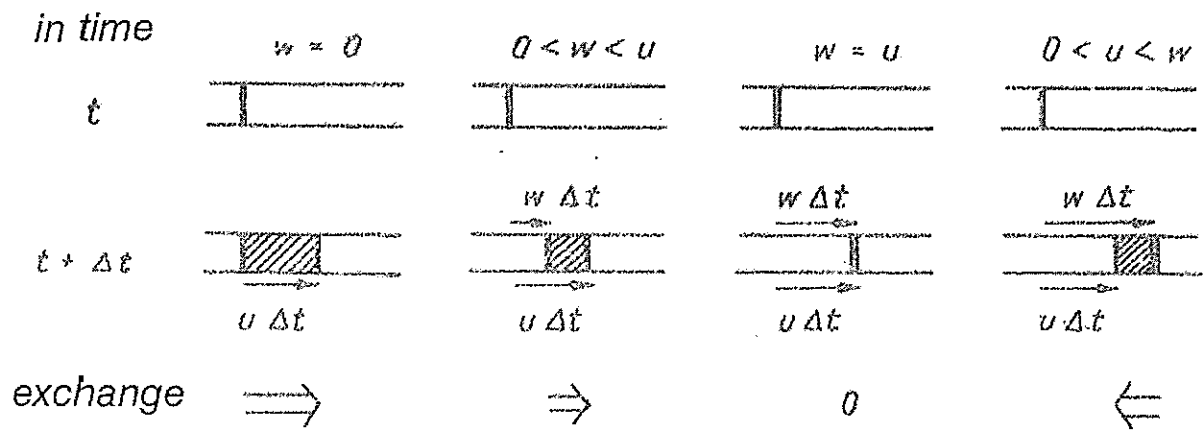


Fig.4.4. The influence of the capture effect on fluid output measurement. The fluid is flowing with the velocity u through the control cross-section (thick line) moving with velocity w along the pipeline. The crossed area corresponds to the fluid mass exchanged during the time Δt

$$(4.22a) \quad \vec{J}_{\vec{w}} = - \oint_{\vec{s}_{\vec{w}}} (d^2\vec{s} \cdot \vec{H}_{\vec{w}}) = - \iiint_V \operatorname{div} \vec{H}_{\vec{w}} d^3V$$

$$(4.22b) \quad \vec{J}_{\vec{w}} = - \oint_{\vec{s}_{\vec{w}}} (d^2\vec{s} \cdot \vec{T}_{\vec{w}}) = - \iiint_V \operatorname{div} \vec{T}_{\vec{w}} d^3V$$

Fig. 4.5 In case of a scalar EQ (Fig. 4.5), on those parts of the shell $\vec{s}_{\vec{w}}$ where $d^2\vec{s}$ and $\vec{H}_{\vec{w}}$ form an obtuse angle, or where $(d^2\vec{s} \cdot \vec{H}_{\vec{w}}) < 0$, the transfer is positive-valued, being an inflow into the region and increasing the storage of EQ. At an acute angle, or at $(d^2\vec{s} \cdot \vec{H}_{\vec{w}}) > 0$, the outflow to the surroundings takes place, and at the right angle, or at $(d^2\vec{s} \cdot \vec{H}_{\vec{w}}) = 0$, there is no transfer at all. The given angle criterion is, of course, based on the principle that the shell is outwardly bound with respect to enclosed region.

The interpretation of the alternative form of Eq.(4.22a) is borrowed from the fluid mechanics. In the places where $\operatorname{div} \vec{H}_{\vec{w}} > 0$ the transfer-flux of a scalar EQ arises, and where $\operatorname{div} \vec{H}_{\vec{w}} < 0$ it vanishes. Within the region of satisfying the equation $\operatorname{div} \vec{H}_{\vec{w}} = 0$, the transfer-flux is composed of sourceless tubes. The transfer of a scalar EQ through cross-section of any segment of such a tube is the same (e.g. the rate of flow of incompressible fluid).

The angle criterion cannot be applied to the classification of transfer of a vector EQ, since in this case the flux density is a tensor.

The transfer of EQ between the region and its surroundings is a manifestation of physical interaction occurring across the boundary

Fig. 4.6 Consider two adjacent regions numbered (1) and (2), and divided by the interface s (Fig. 4.6). We replace the interface with two adhering surfaces s_1 and s_2 , each of them being a part of shell closing the region (1) or (2), respectively. Since the shells are outwardly bound, the adhering surface elements are opposite vectors: $d^2\vec{s}_1 + d^2\vec{s}_2 = 0$. If the density fields \vec{H} and \vec{T} are continuous, the

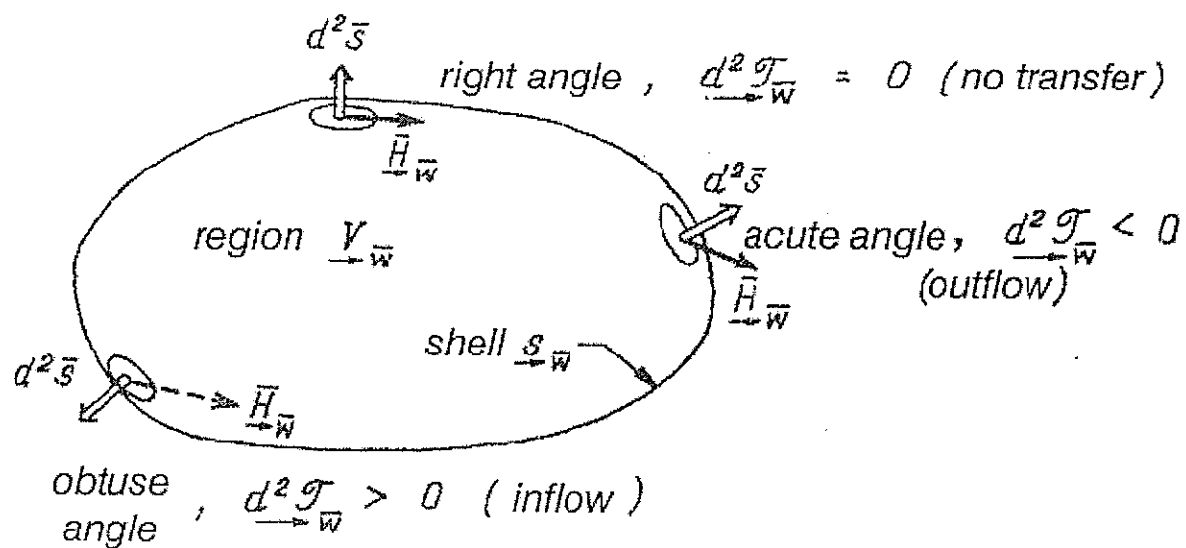


Fig. 4.5. Angle criterion of extensive scalar quantity exchange

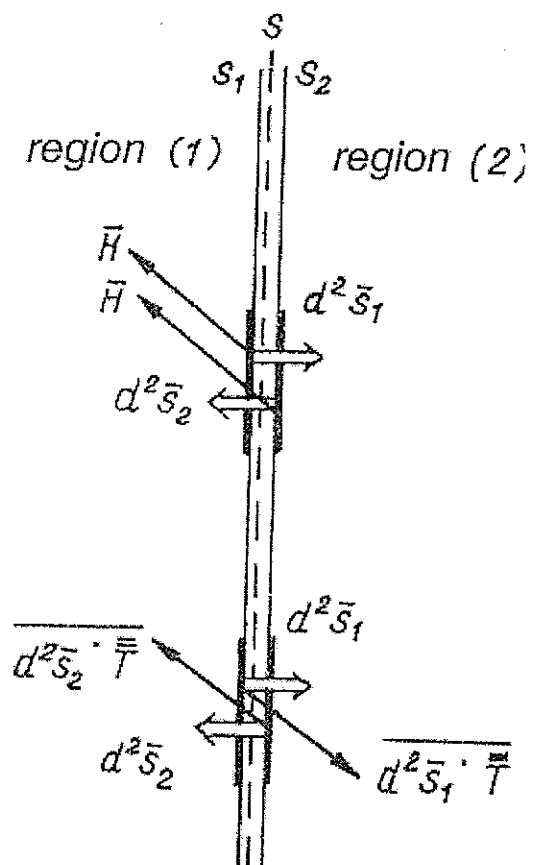


Fig.4.6. Surface interaction between two regions

must be identically-valued on either adhering surface element. Thus

$$(4.23a) \quad (d^2 \bar{s}_1 \bar{H}) = - (d^2 \bar{s}_2 \bar{H}), \quad (4.23b) \quad \overline{d^2 \bar{s}_1 \bar{T}} = - \overline{d^2 \bar{s}_2 \bar{T}}$$

Assuming the following rule for the sign of action

$$(4.24) \quad \left\{ \begin{array}{l} \text{The elementary surface action} \\ \text{exerted by region (1) on region (2)} = (d^2 \bar{s}_1 \bar{H}), \quad \overline{d^2 \bar{s}_1 \bar{T}} \\ \text{exerted by region (2) on region (1)} = (d^2 \bar{s}_2 \bar{H}), \quad \overline{d^2 \bar{s}_2 \bar{T}} \end{array} \right.$$

we find that the relations (4.23) express the transport version of the law of action and reaction. Eq.(4.23a), as applied to the mass (which is a scalar EQ), explains that the substance flow out of the region (1) is equal to the flow into the region (2). If Eq.(4.23b) concerns the momentum (which is a vector EQ), then \bar{T} stands for the stress, and $\overline{d^2 \bar{s}_1 \bar{T}}$, $\overline{d^2 \bar{s}_2 \bar{T}}$ express two opposite surface forces of interaction between the regions (1) and (2), in compliance with the Third Law of Dynamics.

The rule (4.24) is still valid when applied to any finite surface being in motion. Thus, in case of the shell \bar{s}_w of a referential balance region, we have

$$(4.25a) \quad + \oint_{\bar{s}_w} (d^2 \bar{s} \bar{H}_w), + \oint_{\bar{s}_w} \overline{d^2 \bar{s} \bar{T}_w} \quad \begin{array}{l} \text{the surface action exerted by} \\ \text{the balance region on surroundings} \end{array}$$

$$(4.25b) \quad - \oint_{\bar{s}_w} (d^2 \bar{s} \bar{H}_w), - \oint_{\bar{s}_w} \overline{d^2 \bar{s} \bar{T}_w} \quad \begin{array}{l} \text{the surface action exerted by} \\ \text{surroundings on the balance region} \end{array}$$

Comparing Eqs.(4.22a,b) and (4.25b) we find that the transfer, as a balance term, is the surface action exerted by surroundings on the balance region.

4.8. The Momentum Transfer as the Surface Force

From the phenomenological viewpoint, the so-called non-convective transfer of momentum (see Sec. 5.7) is identified with the surface force occurring on the shell of balance region. The non-convective

transfer-flux density of momentum is the stress tensor $\bar{\bar{\Pi}}$. It may take specific shapes of the elastic stress $\bar{\bar{\Pi}}_e$, the viscous stress $\bar{\bar{\Pi}}_v$, or the pressure stress $\bar{\bar{\Pi}}_p = \bar{I}p$, where p denotes the scalar pressure.

Long time ago the hydrostatics defined the pressure p so that

$$(4.26a) \quad \oint_S p \, d^2\bar{s} = \oint_S \overline{d^2\bar{s} \cdot \bar{\bar{\Pi}}_p} \quad \begin{array}{l} \text{expresses the pressure force} \\ \text{(or thrust) exerted by the} \\ \text{balance region (filled with} \\ \text{fluid) on its surroundings} \end{array}$$

(the equivalence of both forms results from the rule 1.18d and substitution of $\bar{\bar{\Pi}}_p = \bar{I}p$).

Some time later the elasticity theory introduced the concept of the elastic stress tensor $\bar{\bar{\Pi}}'_e$ such that

$$(4.26b) \quad \oint_S \overline{d^2\bar{s} \cdot \bar{\bar{\Pi}}'_e} \quad \begin{array}{l} \text{expresses the elastic force exerted by the} \\ \text{surroundings (e.g. a load) on the balance} \\ \text{region (e.g. a stretched bar)} \end{array}$$

On the analogy of $\bar{\bar{\Pi}}'_e$, the fluid mechanics defined the viscous stress $\bar{\bar{\Pi}}'_v$ so that

$$(4.26c) \quad \oint_S \overline{d^2\bar{s} \cdot \bar{\bar{\Pi}}'_v} \quad \begin{array}{l} \text{expresses the viscous force exerted by} \\ \text{the surroundings on the balance region} \\ \text{(filled with fluid)} \end{array}$$

From the definitions (4.26a,b,c) and the sign convention (4.25) we find that

$$(4.26d) \quad \left\{ \begin{array}{l} \text{the non-convective} \\ \text{transfer-flux} \\ \text{density of momentum} \end{array} \right\} \begin{array}{l} \text{in the elastic body} = - \bar{\bar{\Pi}}'_e \\ \text{in the fluid} = + \bar{I}p - \bar{\bar{\Pi}}'_v \end{array}$$

Being a result of fortuitous conventions, the nonuniformity of signs which precede different stresses makes the notation of many equations deprived of consistency and elegance. In the balance equations of all EQs save the momentum, the divergence term is preceded by the minus sign. In all the phenomenological relations save Hooke's elasticity law and Newton's viscosity law, the gradient of the thermodynamic force is preceded by the minus sign.

In order to get rid of this inconsequence, we shall follow some

few authors (e.g. Bird, Stewart & Lightfoot, see Bibliography 3) who assume the uniform sign convention for the resultant stress $\bar{\bar{\Pi}}$, the elastic stress $\bar{\bar{\Pi}}_e$, and the viscous stress $\bar{\bar{\Pi}}_v$. Since this convention is opposite to the traditional one, we have

$$(4.26e) \quad \bar{\bar{\Pi}} = -\bar{\bar{\Pi}}', \quad \bar{\bar{\Pi}}_e = -\bar{\bar{\Pi}}'_e, \quad \bar{\bar{\Pi}}_v = -\bar{\bar{\Pi}}'_v$$

Identifying the resultant stress $\bar{\bar{\Pi}}$ with the non-convective transfer-flux density of momentum, and basing on the rule (4.25a), we find that

$$(4.27a) \quad \oint_{\vec{s} \rightarrow \vec{w}} \overline{d^2 \vec{s} \cdot \bar{\bar{\Pi}}} \quad \text{expresses the surface force exerted by the substantial region on its surroundings}$$

$$(4.27b) \quad \bar{\bar{\Pi}} = \bar{\bar{\Pi}}_e \quad \text{for the elastic body}$$

$$(4.27c) \quad \bar{\bar{\Pi}} = \bar{\bar{\Pi}}_p + \bar{\bar{\Pi}}_v \quad \text{for the fluid}$$

4.9. The Basic Balances of a Continuous Extensive Quantity

Substituting Eqs.(4.7), (4.8) and (4.22) into the basic balances (4.4), we obtain universally-valid integral balances of the continuous scalars and vector EQ, for a referential region:

$$(4.28a) \quad \frac{d}{dt} \int_{\vec{V} \rightarrow \vec{w}} \psi \, d^3V = \int_{\vec{V} \rightarrow \vec{w}} \Psi \, d^3V - \oint_{\vec{s} \rightarrow \vec{w}} (d^2 \vec{s} \cdot \vec{H}_{\vec{w}})$$

$$(4.28b) \quad \underbrace{\frac{d}{dt} \int_{\vec{V} \rightarrow \vec{w}} \vec{P} \, d^3V}_{\text{rate of change of storage}} = \underbrace{\int_{\vec{V} \rightarrow \vec{w}} \vec{R} \, d^3V}_{\text{production}} - \underbrace{\oint_{\vec{s} \rightarrow \vec{w}} \overline{d^2 \vec{s} \cdot \vec{T}}_{\vec{w}}}_{\text{transfer}}$$

Substituting $\vec{w} = 0$ and using the rules (3.1e) and (4.21), we obtain the integral balances for an immobile region:

$$(4.29a) \quad \frac{\partial}{\partial t} \int_V \psi \, d^3V = \int_V \Psi \, d^3V - \oint_S (d^2 \vec{s} \cdot \vec{H})$$

$$(4.29b) \quad \underbrace{\frac{\partial}{\partial t} \int_V \vec{P} \, d^3V}_{\text{rate of change of storage}} = \underbrace{\int_V \vec{R} \, d^3V}_{\text{production}} - \underbrace{\oint_S \overline{d^2 \vec{s} \cdot \vec{T}}}_{\text{transfer}}$$

Since they deal with finite regions, the integral balances (4.28) and (4.29) are mixed differential-integral equations. They have, however, their counterparts being purely differential equations.

In order to obtain those equations we transform the balances (4.28) and (4.29) as follows: to the left-hand side we apply the Leibniz-Reynolds transformation (3.30), and in the term of transfer we use the GGO theorem (1.34). Shifting all the terms to one side we obtain a purely volume integral which is equal to zero for arbitrarily small region, because the fields concerned are assumed continuous. This way the integrated function must be zero everywhere. It has a shape of purely differential equation which links up the densities of EQ as the field (intensive) quantities. Shifting the terms concerning production and transfer to the right-hand side, we obtain the differential balance of EQ which is referred no more to any region but to a point in space. It must be emphasized that the left-hand side of a differential balance (corresponding to the rate of change of storage) is not, in general, a time-derivative but a Reynoldsian.

Starting from Eqs.(4.28) and following the outlined procedure, we derive the differential balances of the scalar and vector EQ, for a referential point:

$$(4.30a) \quad \text{reyn}_{\vec{w}} \psi = \frac{d\psi}{dt} \vec{w} + \psi \operatorname{div} \vec{w} = \psi - \operatorname{div} \vec{H}_{\vec{w}}$$

$$(4.30b) \quad \underbrace{\text{reyn}_{\vec{w}} \bar{P} = \frac{d\bar{P}}{dt} \vec{w} + \bar{P} \operatorname{div} \vec{w}}_{\text{rate of change of storage}} = \underbrace{\bar{R}}_{\text{production}} - \underbrace{\operatorname{div} \vec{T}_{\vec{w}}}_{\text{transfer}}$$

Since \vec{w} is arbitrary, the equations given are of universal character. Substituting $\vec{w} = 0$ into Eqs.(4.30), or applying the above-described procedure to the integral balances (4.29), we obtain the differential balances for the particular case of an immobile point

$$(4.31a) \quad \text{reyn } \Psi = \frac{\partial \Psi}{\partial t} = \Psi - \text{div } \bar{H}$$

$$(4.31b) \quad \underbrace{\text{reyn } \bar{P}}_{\text{rate of change of storage}} = \underbrace{\frac{\partial \bar{P}}{\partial t}}_{\text{production}} = \underbrace{\bar{P} - \text{div } \bar{T}}_{\text{transfer}}$$

4.10. Theorem on the Equivalence of Balances

If any of 4 types of the basic balances (for the referential/immobile region/point) is valid with respect to a specific continuous EQ, then the 3 remaining balances keep their validity with respect to the same EQ. This is somewhat simplified form of the theorem on equivalence of balances. It is applicable to the balances of either scalar or vector EQ, i.e. to the group of Eqs.(4.28a), (4.29a), (4.30a), (4.31a), or Eqs.(4.28b), (4.29b), (4.30b), (4.31b). The theorem can be proved for 12 possible twin combinations in each group, following indications given in Sec. 4.9. The exercises 4.1 and 4.2 contain two exemplifying proofs.

The discussed theorem is of immense use as a universal recipe. If, basing on the physical premises, some equation (e.g. the Second Law of Motion) can be interpreted as a definite type of balance (e.g. of the momentum), then we are allowed to use the remaining equivalent types of balances, provided the specific forms of the densities of storage, production, and transfer-flux were preserved in each equation. This way we avoid tedious mathematical operations which are so familiar in some of the textbooks of the fluid mechanics and thermodynamics, containing separate transformations of balances for the individual cases of the mass, momentum, energy, etc.

Exercise 4.1. Prove the theorem on equivalence of balances (4.28a) and (4.31a), making no intermediary use of Eq.(4.30a).

Exercise 4.2. Prove the theorem on equivalence of balances (4.28b) and (4.31b), making no intermediary use of Eq.(4.30b).

Exercise 4.3. Deduce the rules (4.20a,b), making no use of the geometric image of the capture effect.

Instructions. We compare the balances (4.30a,b) and (4.31a,b) in the respective pairs, at the instant when the referential and immobile point cover each other (because of the assumed continuity of the fields, the events of such kind fill the time-space interval continuously). Subtracting the differential balances with properly developed Reynoldsians, we obtain equations $\text{div } \bar{\mathbf{B}} = 0$ and $\overline{\text{div}} \bar{\mathbf{b}} = 0$, where $\bar{\mathbf{B}}$ and $\bar{\mathbf{b}}$ are sums of certain vectors and tensors, respectively, appearing in Eqs.(4.30) and (4.31). Because of the rules (1.30d) and (1.31d), the solutions of those equations are curls of arbitrary vector or tensor fields, $\bar{\mathbf{A}}$ or $\bar{\mathbf{a}}$, respectively: $\bar{\mathbf{B}} = \overline{\text{rot}} \bar{\mathbf{A}}$, $\bar{\mathbf{b}} = \overline{\text{rot}} \bar{\mathbf{a}}$. We cannot, however, accept the infinite number of solutions, because, from the physical viewpoint, all the constituents of $\bar{\mathbf{B}}$ and $\bar{\mathbf{b}}$ are univocally definite and experimentally measurable. The only physical solutions result from substitutions $\bar{\mathbf{A}}, \bar{\mathbf{a}} = 0$, and this leads to the sought Eqs.(4.20a,b).

4.11. The Conservative Quantity

Such an EQ which can be neither created nor annihilated anywhere and at any time, we call the conservative quantity. The production of such a quantity must vanish in arbitrarily small time-space interval, or

$$(4.32) \quad \Psi = 0, \quad \bar{\mathbf{R}} = 0 \quad \text{always and everywhere, for conservative EQ}$$

Logically, this postulate should be called the law of conservation of an EQ, and not, as it is notoriously written in many textbooks and papers, the balance equation of any EQ, even a non-conservative one.

Since the postulate (4.32) concerns the production density, it is of the field character. If expressed in an integral form (for a finite region), the law of conservation would be deprived of its

universal character. Indeed, if there would be non-zero creation and annihilation in different parts of the system, their sum (production) remaining always zero, then not the quantity but the system could be named conservative.

The basic balances for the conservative EQs result from substitution of Eq.(4.32) into the balances (4.28), (4.29), (4.30) and (4.31). For a conservative EQ the only possible cause of change of storage is the transfer between the balance region and its surroundings.

It results from the definition of conversion (Sec. 4.1) that the sum of mutually convertible EQs within a closed set must be conservative quantity. In the nonrelativistic conditions the following EQs are conservative: the total mass of all components of substance, the resultant electric charge (as the algebraic sum of the positive and negative charges), and the total energy as the sum of all specific energies.

Of the non-conservative character is any singled-out convertible quantity, such as the mass of a selected component of mixture (subjected to creation or annihilation during chemical reaction), the charge of definite sign (subjected to creation and annihilation resulting from the ionization and recombination, respectively), or a specific energy (e.g. the internal energy, which can be created at the expense of annihilated kinetic energy in the process of the viscous dissipation). According to the Second Law of Thermodynamics, the entropy is subjected to creation only, and therefore is not a conservative quantity.

In the strict sense of the postulate (4.32) neither the momentum nor the angular momentum of mass belong to the conservative quantities. The well-known "laws of conservation of linear and angular momentum" are of limited validity, because they assume the existence of central forces only (and no Lorentz force, for example

and deal with an isolated system as a whole, and not with an arbitrary part of it. It seems, however, that the sum of the mass and field momentums, as well as the sum of the mass and field angular momentums, are conservative quantities in the meaning of postulate (4.32). This assumption has been proved, thus far, for the sum of the mass momentum and the electromagnetic field momentum in particular circumstances.

Exercise 4.4. Formulate the differential balance of (resultant) charge at immobile point, denoting the (storage) density of charge with ϱ_e , and regarding the current density \vec{i} as the transfer-flux intensity of charge through an immobile surface. The charge is a scalar conservative EQ. Check the result with Eq.(4.34) given in the next section.

4.12. The Classical Balance of Electromagnetic Energy

Before we derive the balance of electromagnetic energy as an example of scalar EQ, we have to recollect the necessary minimum knowledge on the electromagnetic field theory. The equations will be written in the Lorentz system of units, which employs the universal physical constant c , or the velocity of light in vacuum.

The basic electromagnetic quantities are:

ϱ_e	charge density,	\vec{i}	current density,
\vec{E}	electric field vector,	\vec{D}	electric displacement,
\vec{H}	magnetic field vector,	\vec{B}	magnetic induction.

The vectors \vec{E} and \vec{i} describe the electric field and the current density, respectively, at an immobile point. If the point of reference moves with the referential velocity \vec{w} , then the electric field and the current density are referential vectors $\vec{E}_{\vec{w}}$ and $\vec{i}_{\vec{w}}$, respectively. The relation between $\vec{E}_{\vec{w}}$ and \vec{E} has been found in Ex. 3.20 (Eq. 3.42); \vec{i} differs from $\vec{i}_{\vec{w}}$ by the capture-flux density $\vec{w}\varrho_e$ (see rule 4.20a). Substituting $\vec{w} = \vec{u}$, we find the dependence of \vec{E} and \vec{i} on

$\vec{E}_{\vec{u}}$ and $\vec{I}_{\vec{u}}$, respectively, the latter symbols referred to the substantial point moving with the barycentric velocity \vec{u} :

$$(4.33a) \quad \vec{E} = \vec{E}_{\vec{u}} - \frac{1}{c} \vec{u} \times \vec{B}, \quad (4.33b) \quad \vec{I} = \vec{I}_{\vec{u}} + \varrho_e \vec{u}$$

The balance of charge (see Ex. 4.4) makes the densities ϱ_e and \vec{I} interrelated

$$(4.34) \quad \frac{\partial \varrho_e}{\partial t} + \text{div } \vec{I} = 0,$$

while the first pair of Maxwell's equations postulates the following interdependence among the vectors \vec{E} , \vec{D} , \vec{H} , \vec{B} , and \vec{I}

$$(4.35a) \quad \text{rot } \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \quad (4.35b) \quad \text{rot } \vec{H} = \frac{1}{c} \left\{ \frac{\partial \vec{D}}{\partial t} + \vec{I} \right\}$$

It is easy to prove with the help of Eqs.(4.35) and (1.30c) that

$$(4.36) \quad \left(\vec{E} \frac{\partial \vec{D}}{\partial t} \right) + \left(\vec{H} \frac{\partial \vec{B}}{\partial t} \right) = - (\vec{E} \vec{I}) - \text{div}(\overline{c\vec{E} \times \vec{H}})$$

We have found an important relation being of the dimension of energy per unit time and volume, which is the dimension of a differential balance of energy as a scalar EQ. Though it contains a divergence term, which may be interpreted as a transfer term, the relation (4.36) is not yet a balance, because its left-hand side does not take the shape of a pure time-derivative.

However, in an isotropic and linear medium, the vectors \vec{D} and \vec{B} are parallel and proportional to \vec{E} and \vec{H} , respectively, so that $\left(\vec{E} \frac{\partial \vec{D}}{\partial t} \right) + \left(\vec{H} \frac{\partial \vec{B}}{\partial t} \right) = \frac{\partial}{\partial t} \left\{ \frac{(\vec{E} \vec{D}) + (\vec{H} \vec{B})}{2} \right\}$. In this particular case, Eq. (4.36) assumes the shape of the classical balance of electromagnetic energy (at an immobile point):

$$(4.37) \quad \underbrace{\frac{\partial}{\partial t} \left\{ \frac{(\vec{E} \vec{D}) + (\vec{H} \vec{B})}{2} \right\}}_{\substack{\text{rate of change of storage} \\ \text{for isotropic and linear medium}}} = \underbrace{- (\vec{E} \vec{I})}_{\text{production}} - \underbrace{\text{div}(\overline{c\vec{E} \times \vec{H}})}_{\text{transfer}}$$

The balance given is founded on the hypothesis that the storage

density of electromagnetic energy W_{em} solely depends on the vectors \vec{E} , \vec{D} , \vec{B} , \vec{H} :

$$(4.38) \quad W_{em} = \frac{(\vec{E} \cdot \vec{D}) + (\vec{H} \cdot \vec{B})}{2}$$

According to this assumption, the electromagnetic energy is located at any place where the electromagnetic field does exist, in a region filled with substance as well as in the vacuum.

4.13. The Hysteretic Annihilation of Electromagnetic Energy

Despite of its simplicity and elegance, the classical balance of electromagnetic energy (4.37) is of little use as applied to transport phenomena. Its validity is restricted to a particular medium in which the hysteresis phenomenon does not occur. Moreover, its form makes impossible to reveal the electromechanical conversion of energy and to distinguish the energy transfer by means of the current conduction and the wave radiation.

In order to obtain a balance valid for any medium, remark that

$$(4.39) \quad (\vec{E} \cdot \frac{\partial \vec{D}}{\partial t}) + (\vec{H} \cdot \frac{\partial \vec{B}}{\partial t}) = \frac{\partial W_{em}}{\partial t} + G_{pm}$$

where W_{em} has been defined by Eq.(4.38), and G_{pm} , called the polarization conversion density, is equal to

$$(4.40) \quad G_{pm} = \frac{1}{2} \left\{ (\vec{E} \cdot \frac{\partial \vec{D}}{\partial t}) - (\vec{D} \cdot \frac{\partial \vec{E}}{\partial t}) + (\vec{H} \cdot \frac{\partial \vec{B}}{\partial t}) - (\vec{B} \cdot \frac{\partial \vec{H}}{\partial t}) \right\}$$

This is, in fact, the hysteretic annihilation of electromagnetic energy for the benefit of the internal energy, and it is going together with the hysteresis of electric and magnetic polarization (so-called "heat of hysteresis"). Considering the magnetic phenomenon only, the time-integral of G_{pm} within the cycle of the hysteresis loop gives the expression $\oint (\vec{H} \cdot d\vec{B}) \geq 0$, which is well-known to the electricians. In a linear medium, of course, the hysteresis does not occur, and consequently $G_{pm} = 0$.

Substituting Eq.(4.39) into Eq.(4.36) we obtain the universally-valid (for any medium) balance of electromagnetic energy:

$$(4.41) \quad \frac{\partial W_{em}}{\partial t} = - G_{pm} - (\vec{E} \cdot \vec{I}) - \text{div}(c \vec{E} \times \vec{H})$$

We shall not discuss this balance before we transform two of its terms for the sake of a clearer interpretation.

4.14. Electromechanical Conversion and Conductive Production of Electromagnetic Energy

At first, we shall make the conversion term $(\vec{E} \cdot \vec{I})$ more specific. It is possible to prove (Ex. 4.5) that

$$(4.42) \quad (\vec{E} \cdot \vec{I}) = (\vec{E}_{\vec{u}} \cdot \vec{I}_{\vec{u}}) + (\vec{F}_{ed} \cdot \vec{u})$$

where \vec{F}_{ed} denotes the density of electrodynamical force exerted on the electrically charged or conducting substance

$$(4.43) \quad \vec{F}_{ed} = \rho_e \vec{E}_{\vec{u}} + \frac{1}{c} \vec{I}_{\vec{u}} \times \vec{B} = \underbrace{\rho_e \vec{E}}_{\text{the Coulomb force}} + \underbrace{\frac{1}{c} \vec{I} \times \vec{B}}_{\text{the Lorentz force}}$$

It is worth to notice that \vec{F}_{ed} is a referential invariant, i.e. it remains identical at any arbitrary velocity \vec{w} of the reference point (Ex. 4.6). This property, however, does not concern separately treated constituents of \vec{F}_{ed} , i.e. the Coulomb and Lorentz forces, because $\vec{E}_{\vec{w}} \neq \vec{E}$, $\vec{I}_{\vec{w}} \neq \vec{I}$, generally.

Since \vec{F}_{ed} describes one of the bulk forces exerted on a body, the power density $(\vec{F}_{ed} \cdot \vec{u})$ will appear as a production term in the balance of kinetic energy (5.43a). Hence we conclude that $(\vec{F}_{ed} \cdot \vec{u})$ concerns the reversible electromechanical conversion between the electromagnetic energy and kinetic energy (occurring, for example, in an electric generator or motor):

$$(4.44) \left. \begin{array}{l} \text{production density of} \\ \text{electromagnetic energy} \end{array} \right\} = - \underbrace{(\vec{F}_{ed} \vec{u})}_{\substack{\text{at the expense} \\ \text{for the benefit} \\ \text{of kinetic energy}}} \gtrless 0$$

$\left. \begin{array}{l} > 0, \text{ obtuse angle} \\ < 0, \text{ acute angle} \end{array} \right\} \text{ between } \vec{F}_{ed}, \vec{u}$

The energetic interpretation of the term $(\vec{E} \vec{i}_{\vec{u}})$ of Eq.(4.42) results from Ohm's law. Though the validity of this law is limited to isotropic and linear medium of constant resistivity κ , nevertheless the conclusions obtained can be qualitatively extended over any medium.

In the phenomenological electromagnetics, the symbol \vec{E} is understood to be the macroscopically averaged electric field. In the dense substance, the ordered motion of electric charge is influenced by \vec{E} , as well as by the microscopic electric field, provided it is non-homogenous. The latter influence is phenomenologically described by the motive vector \vec{K} representing the electrochemical and thermoelectric "forces".

For a conductor moving with the barycentric velocity \vec{u} , Ohm's law assumes the following field shape:

$$(4.45) \quad \kappa \vec{i}_{\vec{u}} = \vec{E}_{\vec{u}} + \vec{K}_{\vec{u}}$$

It has to be emphasized that Ohm's law written as $\kappa \vec{i} = \vec{E} + \vec{K}$ remains valid for the resting conductor only (at $\vec{u} = 0$). This statement is essential to the magnetohydrodynamics (see Eq.3.39b), though it is notoriously omitted in most textbooks on electromagnetics.

It results from Ohm's law that in case of an isotropic and linear medium the term of the conductive production of electromagnetic energy consists of two parts:

$$(4.46) \quad \underbrace{-(\vec{E} \cdot \vec{I})}_{\vec{u} \rightarrow \vec{u}} = \underbrace{(\vec{K} \cdot \vec{I})}_{\vec{u} \rightarrow \vec{u}} - \underbrace{\kappa(\vec{I}^2)}_{\vec{u} \rightarrow \vec{u}}$$

conductive production of electromagnetic energy
reversible (≥ 0) electrochemical and thermoelectric conversion
irreversible (< 0) annihilation (Joule's heat or dissipation)

at the expense for the benefit
 > 0 , acute angle
 < 0 , obtuse angle
between $\vec{K}_{\vec{u}}$, $\vec{I}_{\vec{u}}$

of internal energy

In any arbitrary medium, the term $(\vec{E} \cdot \vec{I})_{\vec{u} \rightarrow \vec{u}}$ describes the conversion between electromagnetic energy and internal energy, its both parts, the reversible and the irreversible, being included.

Exercise 4.5. Derive Eq.(4.42).

Instruction. Develop $(\vec{E} \cdot \vec{I})$ with the use of substitutions (4.33), rules (1.19), and denotation (4.43).

Exercise 4.6. Prove that \vec{F}_{ed} is a referential invariant.

Instructions. Substitute Eq.(3.42) and the relation between \vec{I} and $\vec{I}_{\vec{w}}$ (obtained with the help of rule 4.20a) into the latter right-hand side of Eq.(4.43). It will become evident that \vec{F}_{ed} remains unchanged for any arbitrary \vec{w} , including $\vec{w} = 0$, hence $\vec{F}_{ed} = \text{inv} \langle \vec{w} \rangle$.

4.15. The Improved Balance of Electromagnetic Energy

In the classical balance (4.37), the transfer of electromagnetic energy is described by the divergence of Poynting's vector $c\vec{E} \times \vec{H}$. This simple representation, though absolutely faultless from the formal point of view, has not too much convincing physical interpretation. In the crossed static fields, for example, we have $\vec{E} \times \vec{H} \neq 0$, though any flow of energy seems incredible. On the other hand, in such conditions it is always $\text{div } \vec{E} \times \vec{H} = 0$. Thus, if the Poynting vector gives the correct results in computing the transfer through a shell as a whole, its general interpretation as the transfer-flux density is rather dubious.

From the viewpoint of transport phenomena, a more suitable expression for the transfer is based on the concepts of electromagnetic potentials, the scalar one φ , and the vector one \vec{A} . In the final result we obtain two kinds of transfer of electromagnetic energy, characterized by the following flux densities:

$$(4.47a) \quad \vec{J}_{ec} = \varphi \vec{i}$$

conductive transfer

$$(4.47b) \quad \vec{J}_{er} = \varphi \frac{\partial \vec{D}}{\partial t} + \vec{H} \times \frac{\partial \vec{A}}{\partial t}$$

radiative transfer

The conductive transfer can occur in the conductors only, where $\vec{i} \neq 0$. Integrating the vector $-\varphi \vec{i}$ over the shell containing a segment of conductor, we obtain the voltage drop multiplied by the current, i.e. the applied electric power.

The radiative transfer can occur in any medium (also in the vacuum), provided the electromagnetic field keeps its time-variability. As a matter of fact, this is the radiation of the radio waves. This kind of transfer is absent in the stationary and static fields (e.g. at the direct current, or static distribution of constant charges).

Having a convincing physical interpretation, the vectors \vec{J}_{ec} and \vec{J}_{er} are formally correct as well. Though $\vec{J}_{ec} + \vec{J}_{er} \neq c \vec{E} \times \vec{H}$, it is possible to show that

$$(4.48) \quad \text{div}(\vec{J}_{ec} + \vec{J}_{er}) = \text{div}(c \vec{E} \times \vec{H})$$

(the proof has been omitted as too lengthy).

Making use of the transformation (4.48) and previously obtained Eq.(4.42) we are able to discover the full physical interpretation of the intermediate Eq.(4.41). This way we obtain a universal (valid for any medium) relation which has been named the improved balance of electromagnetic energy by the present author:

$$(4.49) \quad \underbrace{\frac{\partial W_{em}}{\partial t}}_{\text{rate of change of storage}} = - \underbrace{G_{pm} - (\vec{E}_{\vec{u}} \vec{i}_{\vec{u}})}_{\substack{\text{production} \\ \text{at expense/for benefit} \\ \text{of} \\ \text{internal energy}}} - \underbrace{(\vec{F}_{ed} \vec{u})}_{\text{kinetic energy}} \underbrace{- \operatorname{div} \vec{J}_{ec} - \operatorname{div} \vec{J}_{er}}_{\substack{\text{transfer through} \\ \text{immobile surface} \\ \text{conductive} \quad \text{radiative}}}$$

5. The Balances of Substantial Quantities

5.1. The Balance of Mass

The mass is a scalar EQ. Its storage density ϱ has been defined by Eq.(2.5a). According to definition (2.6a), the phenomenological motion of mass is characterized by the barycentric velocity \vec{u} . Since any substantial surface $\vec{s}_{\vec{u}}$ moves with the same velocity (Sec. 3.3), the transfer-flux density of mass with respect to such a surface must be zero:

$$(5.1a) \quad \vec{j}_{\vec{u}} = 0$$

Using this postulate in Eq.(4.20a) at $\vec{w} = \vec{u}$, we find that the mass-flux density through immobile surface is

$$(5.1b) \quad \vec{j} = \varrho \vec{u}$$

Applying this once again to the rule (4.20a), we obtain the mass-flux density through referential surface:

$$(5.1c) \quad \vec{j}_{\vec{w}} = \varrho(\vec{u} - \vec{w})$$

Substitution of $\varphi = \varrho$, $\Psi = 0$ (the mass is a conservative quantity), and $\vec{H}_{\vec{w}} = \vec{j}_{\vec{w}}$ into the basic balances (4.28a) and (4.30a) gives the referential balances of mass:

$$(5.2a) \quad \frac{d}{dt} \int_{\vec{V}_{\vec{w}}} \varrho \, d^3V = - \oint_{\vec{S}_{\vec{w}}} (d^2\vec{s} \{ \vec{u} - \vec{w} \}) \varrho$$

$$(5.2b) \quad \operatorname{reyn}_{\vec{w}} \varrho = \frac{d\varrho}{dt} \vec{w} + \varrho \operatorname{div} \vec{w} = - \operatorname{div}(\{ \vec{u} - \vec{w} \} \varrho)$$

Substituting $\vec{w} = \vec{u}$ and $\vec{w} = 0$ into Eq.(5.2a), we obtain the

integral balances of mass for the substantial and immobile region, respectively:

$$(5.3a) \quad \frac{d}{dt} \int_V \rho \, d^3v = 0, \quad (5.3b) \quad \frac{\partial}{\partial t} \int_V \rho \, d^3v = - \oint_S (d^2s \, \bar{u}) \rho$$

The balance (5.3a) asserts that the storage of mass in the substantial region remains constant in time.

Substituting $\bar{w} = \bar{u}$ and $\bar{w} = 0$ into Eq.(5.2b), we obtain the differential balances of mass at the substantial and immobile point, respectively:

$$(5.4a) \quad \text{reyn}_{\rightarrow \bar{u}} \rho = 0, \quad \text{or } (5.4b) \quad \frac{d\rho}{dt} \rightarrow \bar{u} + \rho \, \text{div} \, \bar{u} = 0,$$

$$\text{or } (5.4c) \quad \frac{dv}{dt} \rightarrow \bar{u} = v \, \text{div} \, \bar{u}, \quad v = 1/\rho$$

$$(5.4d) \quad \frac{\partial \rho}{\partial t} + \text{div}(\rho \bar{u}) = 0$$

Eq.(5.4a) asserts that the substantial Reynoldsian of mass density is equal to zero always and everywhere. The alternative Eq. (5.4c) is widely used in the thermodynamics because of the specific volume v contained. Eq.(5.4d) is known as the continuity equation, the name being not too logical since apart from the mass many other EQs are regarded to be continuous.

It results from the balance (5.4d) that

$$(5.5a) \quad \text{div}(\rho \bar{u}) = 0 \quad \text{for stationary flow}$$

in which any local time-changes do not occur ($\partial/\partial t = 0$).

In an incompressible liquid, the mass density field is stationary and homogenous, $\rho = \text{const} \langle t, \vec{r} \rangle$. Applying this to the balance (5.4b) we find that the substantial region is not submitted to dilatation (compare Eq.3.12a):

$$(5.5b) \quad \text{div} \, \bar{u} = 0 \quad \text{for incompressible liquid } (\rho = \text{const})$$

Exercise 5.1. Formulate the balance of mass m of a rocket moving with velocity \vec{w} .

Fig. 5.1 Instructions - (Fig. 5.1). We regard the rocket as a stiff referential region $V_{\vec{w}}$. Its shell $S_{\vec{w}}$ consists of the material outer surface of rocket's hull s_g and an imaginary flat surface s_n covering the hull's opening and penetrable to the exhaust gases leaving the engine. The surface s_n can be represented by the vector \vec{s}_n , outwardly bound with respect to $V_{\vec{w}}$. Over the entire s_g there is no difference between the substantial and referential velocity ($\vec{u} = \vec{w}$), while over s_n the absolute velocity of gas $\vec{u}_g \neq \vec{w}$. For the sake of simplification, we assume that the density of gas ρ_g and its relative velocity $\vec{u}_g = \vec{u}_g - \vec{w}$ are homogeneously distributed over the whole surface s_n . All the assumptions given we apply to the balance (5.2a) and denote by m the integral on the left-hand side. The result is:

$$(5.6) \quad \frac{dm}{dt} \vec{w} = - (\vec{s}_n \vec{u}_g) \rho_g$$

5.2. The Substantial Quantity

The substantial quantity, or SQ, as abbreviated, is an extensive quantity which can be stored in the substance and carried with the substance. The SQ need not to be a conservative quantity, and the carrying with substance is not, in general, the only way of SQ's transport.

The momentum and angular momentum are the examples of a vector SQ. To the scalar SQs belong: the entropy, some specific energies, as well as the volume of a region filled with substance. On the other hand, the electromagnetic energy is not a SQ, because it may be stored even in the vacuum.

Since the mass is a measure of the size of SQ, the storage of SQ depends on the mass stored within the balance region. The continuous SQs are characterized by the mass-densities, or, according to tradition, "specific quantities" (e.g. the specific volume):

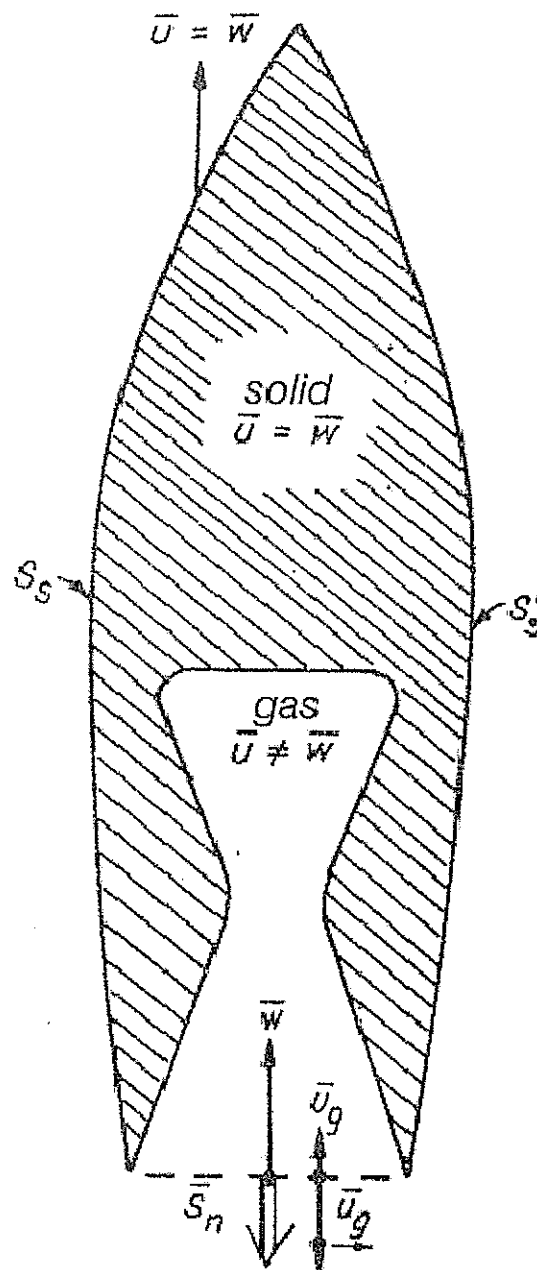


Fig. 5.1. The scheme of a rocket as referential region

Category of the mass-density	Definition	Denotation for	
		scalar SQ	vector SQ
(5.7) $\left. \begin{array}{l} \text{storage} \\ \text{mass-density} \\ \text{of SQ} \end{array} \right\}$	$= \limf \frac{\text{stored SQ}}{\text{stored mass}}$	ε	\bar{Q}
(5.8) $\left. \begin{array}{l} \text{production} \\ \text{mass-density} \\ \text{of SQ} \end{array} \right\}$	$= \limf \frac{\text{produced SQ}}{\text{time} \cdot \text{stored mass}}$	μ	\bar{S}

Comparing definitions (2.5a), (4.5), (4.6), (5.7), and (5.8) we obtain the relations between the volume-densities and the mass-densities of SQs:

$$(5.9a) \quad \Psi = \varrho \varepsilon, \quad (5.9b) \quad \bar{P} = \varrho \bar{Q}, \quad (5.10a) \quad \psi = \varrho \mu, \quad (5.10b) \quad \bar{R} = \varrho \bar{S}$$

We shall use frequently the following transformations of the substantial Reynoldsians (see Ex. 5.2)

$$(5.11a) \quad \underset{\rightarrow \bar{u}}{\text{reyn}}(\varrho \varepsilon) = \frac{\partial(\varrho \varepsilon)}{\partial t} + \text{div}(\bar{u} \varepsilon \varrho) = \frac{d(\varrho \varepsilon)}{dt} \rightarrow \bar{u} + \varrho \varepsilon \text{div} \bar{u} = \varrho \frac{d\varepsilon}{dt} \rightarrow \bar{u}$$

$$(5.11b) \quad \underset{\rightarrow \bar{u}}{\text{reyn}}(\varrho \bar{Q}) = \frac{\partial(\varrho \bar{Q})}{\partial t} + \overline{\text{div}}\{\bar{u} \cdot \bar{Q} \varrho\} = \frac{d(\varrho \bar{Q})}{dt} \rightarrow \bar{u} + \varrho \bar{Q} \text{div} \bar{u} = \varrho \frac{d\bar{Q}}{dt} \rightarrow \bar{u}$$

and the substantial Reynolds' transformations (see Ex. 5.3)

$$(5.12a) \quad \frac{d}{dt} \rightarrow \bar{u} \iint_{\substack{V \\ \rightarrow \bar{u}}} \varrho \varepsilon \, d^3V = \iint_{\substack{V \\ \rightarrow \bar{u}}} \varrho \frac{d\varepsilon}{dt} \rightarrow \bar{u} \, d^3V, \quad (5.12b) \quad \frac{d}{dt} \rightarrow \bar{u} \iint_{\substack{V \\ \rightarrow \bar{u}}} \varrho \bar{Q} \, d^3V = \iint_{\substack{V \\ \rightarrow \bar{u}}} \varrho \frac{d\bar{Q}}{dt} \rightarrow \bar{u} \, d^3V$$

Exercise 5.2. Prove the transformations (5.11a,b).

Instruction. Substitute Eqs.(5.9a,b) into the expressions (3.25a,b) for the Reynoldsians at $\bar{w} = \bar{u}$, develop the derivatives of the products, and use the balance (5.4b).

Exercise 5.3. Prove the transformations (5.12a,b).

Instruction. Substitute the relations (5.11a,b) into the transformations (3.30a,b) at $\bar{w} = \bar{u}$.

5.3. Convective and Non-Convective Transfer of a Substantial Quantity

Substituting Eqs.(5.9) into Eqs.(4.20) at $\bar{w} = \bar{u}$ and rearranging the terms, we obtain the classifying decomposition of the transfer-flux density of a SQ with respect to the immobile surface:

$$\begin{aligned} (5.13a) \quad \bar{H} &= \underbrace{\bar{H}_{\rightarrow \bar{u}}}_{\text{non-convective}} + \bar{u} \varepsilon \varrho \\ (5.13b) \quad \bar{T} &= \underbrace{\bar{T}_{\rightarrow \bar{u}}}_{\text{non-convective}} + \underbrace{\bar{u} \bar{Q} \varrho}_{\text{convective}} \end{aligned}$$

In the convective transfer, the SQ is being carried with substance flowing with the barycentric velocity \bar{u} . This is the capture of SQ's storage by the mobile substantial surface $S_{\bar{u}}$.

According to Eqs.(4.17), at $\bar{w} = \bar{u}$ and with Eqs.(5.9) substituted, there is no convective transfer through the surface to which the barycentric velocity field \bar{u} is tangent:

$$(5.14a) \quad (d^2 \bar{s} \bar{u}) \varepsilon \varrho = 0, \quad (5.14b) \quad (d^2 \bar{s} \bar{u}) \bar{Q} \varrho = 0 \quad \text{at } d^2 \bar{s} \perp \bar{u}$$

The non-convective transfer is directly referred to the substantial surface, so it has nothing in common with the motion of substance itself. The laws governing the non-convective transfer are directly independent of the barycentric velocity. For example, in a body with definite distribution of the temperature field, the heat conduction process (the non-convective transfer of internal energy) will be the same at any velocity of the body. Though the non-convective transfer of momentum due to viscosity depends on the gradient of velocity \bar{u} , nevertheless it does not depend on the velocity itself, and it would remain unaltered by the addition of any homogenous velocity field to \bar{u} .

The flux densities of the non-convective transfer of momentum and energy are of a very frequent use in the fluid mechanics and thermodynamics. Therefore the fully formalized denotations $\bar{H}_{\rightarrow \bar{u}}$ and $\bar{T}_{\rightarrow \bar{u}}$ will appear in the general theory of SQs only, and in the

specific cases we shall replace them with traditional symbols deprived of arrows, e.g. \vec{J}_{tc} (flux density of heat conduction) or $\vec{\Pi}$ (the stress), bearing in mind that they describe the transfer referred to the substantial surface but not the immobile one.

Substituting Eqs.(5.9) into Eqs.(4.20) and using Eqs.(5.13), we obtain the transfer-flux densities of SQs through a referential surface:

$$(5.15a) \quad \vec{H}_{\vec{w}} = \vec{H}_{\vec{u}} - (\vec{w}-\vec{u})\varepsilon\varrho, \quad (5.15b) \quad \vec{T}_{\vec{w}} = \vec{T}_{\vec{u}} - \overline{(\vec{w}-\vec{u})' \vec{Q}} \varrho$$

5.4. The Basic Balances of a Substantial Quantity

Substituting Eqs.(5.9), (5.10) and (5.15) into Eqs.(4.28) and (4.30), we obtain the referential balances of SQs:

$$(5.16a) \quad \frac{d}{dt} \vec{w} \iiint_{V_{\vec{w}}} \varrho \varepsilon d^3v = \iiint_{V_{\vec{w}}} \varrho \mu d^3v - \oint_{S_{\vec{w}}} (d^2s \{ \vec{H}_{\vec{u}} - (\vec{w}-\vec{u})\varepsilon\varrho \})$$

$$(5.16b) \quad \frac{d}{dt} \vec{w} \iiint_{V_{\vec{w}}} \varrho \vec{Q} d^3v = \iiint_{V_{\vec{w}}} \varrho \vec{S} d^3v - \oint_{S_{\vec{w}}} d^2s \{ \vec{T}_{\vec{u}} - \overline{(\vec{w}-\vec{u})' \vec{Q}} \varrho \}$$

$$(5.17a) \quad \text{reyn}(\varrho \varepsilon)_{\vec{w}} = \frac{d(\varrho \varepsilon)}{dt} \vec{w} + \varrho \varepsilon \text{div } \vec{w} = \varrho \mu - \text{div } \vec{H}_{\vec{u}} + \text{div} \{ (\vec{w}-\vec{u})\varepsilon\varrho \}$$

$$(5.17b) \quad \overline{\text{reyn}(\varrho \vec{Q})}_{\vec{w}} = \frac{d(\varrho \vec{Q})}{dt} \vec{w} + \varrho \vec{Q} \text{div } \vec{w} = \varrho \vec{S} - \overline{\text{div } \vec{T}_{\vec{u}}} + \overline{\text{div} \{ (\vec{w}-\vec{u})' \vec{Q} \varrho \}}$$

Substituting $\vec{w} = \vec{u}$ and using the Reynolds transformations (5.12) we obtain the balances of SQs in a substantial region and at a substantial point:

$$(5.18a) \quad \frac{d}{dt} \vec{u} \iiint_{V_{\vec{u}}} \varrho \varepsilon d^3v = \iiint_{V_{\vec{u}}} \varrho \mu d^3v - \oint_{S_{\vec{u}}} (d^2s \vec{H}_{\vec{u}}) \quad (5.18b) \quad \frac{d}{dt} \vec{u} \iiint_{V_{\vec{u}}} \varrho \vec{Q} d^3v = \iiint_{V_{\vec{u}}} \varrho \vec{S} d^3v - \oint_{S_{\vec{u}}} d^2s \vec{T}_{\vec{u}}$$

$$(5.19a) \quad \varrho \frac{d\varepsilon}{dt} \vec{u} = \varrho \mu - \text{div } \vec{H}_{\vec{u}}, \quad (5.19b) \quad \varrho \frac{d\vec{Q}}{dt} \vec{u} = \varrho \vec{S} - \overline{\text{div } \vec{T}_{\vec{u}}}$$

Substituting $\vec{w} = 0$, we obtain the balances of SQs in an immobile region and at an immobile point, with the distinction of the non-

-convective and convective transfer:

$$(5.20a) \quad \frac{\partial}{\partial t} \iiint_V \varrho \varepsilon \, d^3V = \iiint_V \varrho \mu \, d^3V - \oint_S (d^2s \{ \vec{H}_{\vec{u}} + \vec{u} \varepsilon \varrho \})$$

$$(5.20b) \quad \frac{\partial}{\partial t} \iiint_V \varrho \bar{Q} \, d^3V = \iiint_V \varrho \bar{S} \, d^3V - \oint_S d^2s \{ \vec{T}_{\vec{u}} + \vec{u} \bar{Q} \varrho \}$$

$$(5.21a) \quad \left. \frac{\partial(\varrho \varepsilon)}{\partial t} = \varrho \mu - \operatorname{div}(\vec{H}_{\vec{u}} + \vec{u} \varepsilon \varrho) \right| (5.21b) \quad \frac{\partial(\varrho \bar{Q})}{\partial t} = \varrho \bar{S} - \operatorname{div}(\vec{T}_{\vec{u}} + \vec{u} \bar{Q} \varrho)$$

The above collection of 12 basic balances (from 5.16 to 5.21) consists of the scalar and vector group, numbered with a and b, respectively. The theorem on equivalence of balances (Sec. 4.10) is applicable to either group of 6 equations.

Exercise 5.4. Prove the equivalence of the balance sets (5.19a,b) and (5.21a,b).

Instruction. Use the transformations (5.11).

Exercise 5.5. Derive the balances of mass (5.2a,b), (5.3a,b) and (5.4a,b) from the basic balances of SQ.

Instructions. Consider that the mass is a conservative SQ, specify the densities of storage and production ($\varepsilon = \dots$, $\mu = \dots$) basing on the definitions (5.7) and (5.8), and take into account that the transfer of mass is of convective character only (Eq. 5.1b), in order to define \vec{H} .

5.5. The Classification of Forces

Before we discuss the balance of momentum we need to make a digression on the classification of forces. The grains of substance are changing their momentums under the influence of the force fields which are phenomenologically divided into the long-range and short-range categories (Sec. 2.6).

The long-range force fields, such as the gravitational field or the macroscopic (phenomenologically averaged) electromagnetic field

affect in the same way any of the identical grains contained within small volume element. Therefore the force exerted by the long-range field on the substance is proportional to volume and called the volume or body force. Its spatial distribution is described by the volume-density \bar{F} (body force per unit volume).

The body force is a wider concept than the mass force of gravity, or the charge-current forces of Coulomb and Lorentz. In the phenomenological physics we deal with only two body forces: the already discussed (Eq.4.43) electrodynamic force of density \bar{F}_{ed} , and the gravitational force of density \bar{F}_g . This way the density of the resultant body force is:

$$(5.22) \quad \bar{F} = \bar{F}_{ed} + \bar{F}_g$$

The density of gravitational force is proportional to the gravitational field intensity (or the gravitational acceleration) \bar{g} , the vector \bar{g} being expressed by the gradient of the gravitational potential Γ :

$$(5.23a) \quad \bar{F}_g = \rho \bar{g}, \quad (5.23b) \quad \bar{g} = - \overline{\text{grad}} \Gamma$$

Since the gravity is of attractive character, the potential Γ assumes negative values. In the terrestrial conditions, however, it is convenient to make use of the variable excess potential with respect to a constant negative value. At a proper choice of the constant, the symbol Γ can be regarded as non-negative excessive gravitational potential, depending on the scalar terrestrial acceleration $g = 9,8 \text{ m/s}^2$ and the elevation h above a conventional reference level:

$$(5.24) \quad \Gamma = gh \quad \text{in terrestrial conditions}$$

Neglecting the influence of motion of heavenly bodies (and consequently excluding the tidal motion of oceans from our considerations!), we can take the next step in assuming that the spatial

distribution of the field Γ remains time-invariable in a frame based on earth, or that the field Γ is stationary (see Sec. 3.9):

$$(5.25a) \quad \Gamma = \text{const} \langle t \rangle, \quad \partial \Gamma / \partial t = 0 \quad \text{in terrestrial conditions}$$

Consequently, according to Eqs.(3.3a) and (5.23b), we have

$$(5.25b) \quad \frac{d\Gamma}{dt} \vec{u} = (\vec{u} \cdot \text{grad } \Gamma) = - (\vec{u} \cdot \vec{g}) \quad \text{in terrestrial conditions}$$

The short-range interaction is connected with the intra-atomic bonds and intra-molecular collisions. Since the short-range force fields, or microfields, vanish in the process of phenomenological averaging (Eq.2.13b), the forces themselves cannot be macroscopically represented as vectors proportional to the field intensities. Nevertheless the forces mentioned manifest their existence in the phenomena of elasticity, viscosity, and pressure.

From the viewpoint of microphysics, the short-range interaction may be described as the transfer of momentum between individual grains in the course of oscillations or collisions. Phenomenologically, this is a transport of particular EQ occurring within the entire volume of region, with the resultant effect manifested over the region's shell. Consequently, the short-range interaction has the nature of a surface force (Eq.4.27a). Its tensorial density is called the stress $\vec{\Pi}$. The force exerted on a surface element $d^2\vec{s}$ is equal to $-\vec{\Pi} \cdot d^2\vec{s}$ (the negative sign results from convention 4.25b), and depends on the intrinsic state of the body ($\vec{\Pi}$), as well as on the direction of $d^2\vec{s}$, generally differing from the direction of force.

5.6. The Classification of Stresses

In the phenomenological physics we distinguish three kinds of surface forces due to elasticity, viscosity, and pressure. The first one occurs in the solids, the second and third are peculiar to the fluids. The rheology considers both elasticity and viscosity of the "flowing" solid bodies.

The elastic stress $\bar{\bar{\Pi}}_e$ and the viscous stress $\bar{\bar{\Pi}}_v$ depend on the tensors of deformation $\bar{\bar{g}}\bar{\bar{r}}\bar{\bar{a}}\bar{\bar{d}} \delta\bar{r}$ and deformation rate $\bar{\bar{g}}\bar{\bar{r}}\bar{\bar{a}}\bar{\bar{d}} \bar{u}$ of the substantial region, respectively (see Secs. 3.10, 3.11). In a linear medium with a not too big strain or strain rate, the stresses $\bar{\bar{\Pi}}_e$ and $\bar{\bar{\Pi}}_v$ are governed by the generalized laws of Hooke and Newton:

$$(5.26a) \quad \bar{\bar{\Pi}}_e = -\mu_b \bar{I} \operatorname{div} \delta\bar{r} - \mu_s \bar{\bar{g}}\bar{\bar{r}}\bar{\bar{a}}\bar{\bar{d}} \delta\bar{r} \quad \left\{ \begin{array}{l} \text{Hooke's law for} \\ \text{elastic stress} \\ \mu_b, \mu_s = \text{const} \end{array} \right.$$

$$(5.26b) \quad \bar{\bar{\Pi}}_v = -\eta_b \bar{I} \operatorname{div} \bar{u} - \eta_s \bar{\bar{g}}\bar{\bar{r}}\bar{\bar{a}}\bar{\bar{d}} \bar{u} \quad \left\{ \begin{array}{l} \text{Newton's law for} \\ \text{viscous stress} \\ \eta_b, \eta_s = \text{const} \end{array} \right.$$

Both laws are formally similar. The first terms on the right-hand sides are spherical tensors proportional to dilatation $\operatorname{div} \delta\bar{r}$, or the rate of dilatation $\operatorname{div} \bar{u}$ (compare Eq. 3.34), so they depend on the change of volume of the substantial region. On account of Eq. (5.5b) they vanish in an incompressible medium, for example in the perfect liquid.

Since the second terms are symmetric parts of the deviators, they depend on the pure strain or its rate (see Sec. 3.11). This way, in an ordinary elastic or viscous medium, the stresses are symmetric: $\bar{\bar{\Pi}}_e = \bar{\bar{\Pi}}_e^s$, $\bar{\bar{\Pi}}_v = \bar{\bar{\Pi}}_v^s$ (compare the symbol 1.10a).

It is worth to mention that in the so-called Cosserat elastic medium and in the polar fluid, we have to consider some additional stresses due to the relative rotations of microelements of substance. They cause $\bar{\bar{\Pi}}_e$ and $\bar{\bar{\Pi}}_v$ to become asymmetric tensors (containing also antisymmetric parts).

The proportionality factors in Eqs. (5.26) are called the Lamé's elasticity moduli μ_b , μ_s , and the (dynamic) viscosity coefficients η_b , η_s . The suffix b denotes the bulk, and s the shear, elasticity or viscosity.

In order to preserve the uniformity of all the constitutive equations (such as the phenomenological relations for heat conduction,

diffusion, etc.) the laws (5.26) have been denoted in a shape differing from traditional. For example historically-grounded form of the law of viscosity (5.26a) is as follows:

$$(5.26c) \quad \bar{\Pi}'_V = + \gamma_b \bar{I} \operatorname{div} \bar{u} + 2\gamma'_s \bar{g} \bar{r} \bar{a} \bar{d} \bar{s} \bar{u}$$

where $\bar{\Pi}'_V = - \bar{\Pi}_V$ and $\gamma'_s = \frac{1}{2} \gamma_s$ are traditionally formulated viscous stress and shear viscosity coefficient, respectively. The differing signs result from the opposite definitional conventions for the stress (Eq.4.26e), and the number 2 at γ'_s is due to the fact that the viscosity coefficient has been at first defined for a particular case of flow between parallel plates.

Making use of the so-called first and second viscosity coefficients γ_1, γ_2 , it is possible to present Newton's law (5.26b) in the form of Eq.(5.28) (given in Ex. 5.6), which is more convenient for computing, though less clear in the physical interpretation, because of the blurred role of pure strain.

Fig. 5.2

The viscosity force is a manifestation of the momentum transport due to thermal motions of fluid molecules. Generally, the directions of the resultant transport and the barycentric velocity are different. For example in the ^{plane} ~~flat~~ flow (two-dimensional but unidirectional) of incompressible fluid, the transport of momentum is perpendicular to the momentum itself (Fig. 5.2).

In the fluid, the surface-density of the pressure force is the pressure stress

$$(5.27a) \quad \bar{\Pi}_p = \bar{I} p$$

This is a spherical tensor degenerating to the scalar pressure p . Owing to this property, the pressure force $- \bar{d}^2 \bar{s} \bar{\Pi}_p = - p \bar{d}^2 \bar{s}$ (see the rule 1.18d) exerted on an element of shell $\bar{d}^2 \bar{s}$ is always normal to the surface (compare Eq.4.26a), and the "volume-density" of pressure force may be expressed by the pressure gradient (rule 1.31a):

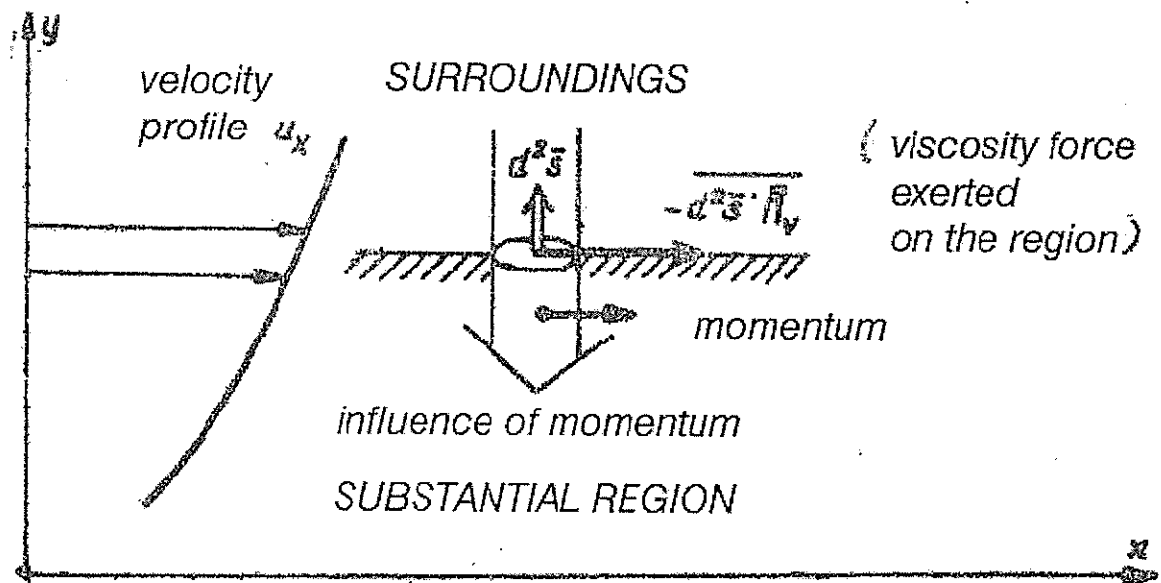


Fig. 5.2. Viscous exchange of momentum in an incompressible fluid plane flow

$$(5.27b) \quad -\overline{\text{div}} \bar{\Pi}_p = -\overline{\text{grad}} p$$

Hence we obtain two alternative interpretations of the pressure force as a term of balance. According to the first interpretation, the pressure force is regarded as the transfer of momentum (volume integral of divergence). In the second case, it is production of momentum (integral of a gradient). Even though the fluid is at phenomenological rest ($\bar{u} = 0$) and contains zero storage of macroscopic momentum, the transport of molecular momentum occurs incessantly ($p \neq 0$), being equalized by the reaction force $p d^2\bar{s}$ on each wall-surface element of the vessel. In a resting liquid, the resultant transport of molecular momentum can be regarded as the production $-\overline{\text{grad}} p$, being locally compensated with the production $\rho \bar{g}$ due to gravitational force.

Exercise 5.6. Prove that the law (5.26b) may be presented in the following form:

$$(5.28) \quad \bar{\Pi}_v = -\gamma_2 \bar{I} \text{div } \bar{u} - \gamma_1 \bar{g} \bar{\text{grad}}^S \bar{u}, \quad \gamma_1 = \gamma_s, \quad \gamma_2 = \gamma_b - \frac{1}{3} \gamma_s$$

Instructions. Transform the symmetric deviator according to the rules (1.13b), (1.9) and (1.29a). Prove that Eqs.(5.28) and (5.26b) become identical in case of the incompressible liquid.

5.7. The Balance of Momentum

Being the product of mass and velocity, the momentum is a vector SQ. According to definitions (5.7) and (5.9b), the momentum of continuous substance has the storage mass-density equal to the barycentric velocity \bar{u} , and the volume-density $\rho \bar{u}$.

In the nonrelativistic conditions, the balance of momentum originates from Newton's Second Law of Motion. It states that the time rate of change of momentum of a body (i.e. momentum stored within the substantial region) is equal to all the forces exerted on the body, that is the resultant body force of density \bar{F} plus the

resultant surface force expressed by the shell integral of stress $\bar{\bar{\bar{\Pi}}}$ (Eq.4.27a with opposite sign):

$$(5.29a) \quad \frac{d}{dt} \int_{V_{\bar{u}}} \rho \bar{u} d^3V = \int_{V_{\bar{u}}} \bar{F} d^3V - \oint_{S_{\bar{u}}} d^2s \cdot \bar{\bar{\bar{\Pi}}}$$

A comparison between the above equation and the basic balance (5.18b) suggests that the body force may be interpreted as the momentum production of density \bar{F} , and the surface force seems to be the non-convective transfer of momentum of the flux density equal to stress $\bar{\bar{\bar{\Pi}}}$. Applying the theorem on equivalence of basic balances (5.18b) and (5.19b) to the Second Law of Motion (5.29a), we obtain the equation of motion of continuous substance:

$$(5.29b) \quad \rho \frac{d\bar{u}}{dt} = \bar{F} - \text{div} \bar{\bar{\bar{\Pi}}}$$

The momentum balances for an immobile region and point result from the theorem on equivalence of basic balances (5.18b), (5.19b), (5.20b), and (5.21b):

$$(5.30a) \quad \frac{\partial}{\partial t} \int_V \rho \bar{u} d^3V = \int_V \bar{F} d^3V - \oint_S d^2s \cdot \left\{ \bar{\bar{\bar{\Pi}}} + \rho \bar{\bar{\bar{u}}} \bar{\bar{\bar{u}}} \right\}$$

$$(5.30b) \quad \frac{\partial(\rho \bar{u})}{\partial t} = \bar{F} - \text{div}(\bar{\bar{\bar{\Pi}}} + \rho \bar{\bar{\bar{u}}} \bar{\bar{\bar{u}}})$$

The terms containing the dyadic of barycentric velocity $\bar{\bar{\bar{u}}} \bar{\bar{\bar{u}}}$ express the convective transfer of momentum. The only places where this kind of transfer does not occur are the surfaces to which the velocity field \bar{u} is tangent (Eq.5.14b).

From the equivalence of basic balances (5.20b), (5.21b), (5.16b) and (5.17b) we obtain the momentum balances for a referential region and point:

$$(5.31a) \quad \frac{d}{dt} \int_{V_{\vec{w}}} \rho \vec{u} \, d^3V = \int_{V_{\vec{w}}} \vec{F} \, d^3V - \oint_{S_{\vec{w}}} d^2s \, \left\{ \vec{n} - \rho(\vec{w}-\vec{u}) \cdot \vec{u} \right\}$$

$$(5.31b) \quad \frac{d(\rho \vec{u})}{dt} \cdot \vec{w} + \rho \vec{u} \operatorname{div} \vec{w} = \vec{F} - \operatorname{div} \vec{n} + \operatorname{div} \left\{ \rho(\vec{w}-\vec{u}) \cdot \vec{u} \right\}$$

It has to be strongly emphasized that Eqs.(5.29a,b) keep their validity with respect to the substantial region and point only. The direct application of the Second Law of Motion to any kind of region, for example to the rocket of variable mass, is erroneous. In such case, the proper way is to start from the referential type of balance (Ex. 5.14).

5.8. Some Applications of the Balance of Momentum

Assuming specific conditions and properties of considered substance in Eqs.(5.29), (5.30) and (5.31), we obtain a variety of particular forms of the momentum balance. Some of them will be shown in the exercises given below.

Among the body forces (5.22), \vec{F}_{ed} has to be considered only in the conducting medium (the windings of electric machines, the magnetohydrodynamic flow), and \vec{F}_g is usually neglected in the gaseous medium (excepting for the cases of large difference in altitude and of the free convection). The specification of stresses in an elastic body and fluid has been given in Eqs.(4.27b,c). At a stationary flow all $\partial/\partial t = 0$, and in a static state, additionally $\vec{u} = 0$.

Exercise 5.7. Motivate the origin and assumptions of the basic equation of elastostatics

$$(5.32a) \quad \rho \vec{g} - \operatorname{div} \vec{n}_e = 0$$

Exercise 5.8. Motivate the origin and assumptions of the basic equation of hydrostatics

$$(5.32b) \quad \rho \vec{g} - \operatorname{grad} p = 0$$

Exercise 5.9. Motivate the origin and assumptions of the Navier-

-Stokes equation for incompressible liquid (instruction: use the rule 1.31c):

$$(5.33a) \quad \rho \frac{d\vec{u}}{dt} = \rho \vec{g} - \text{grad } p + \frac{1}{2} \eta_s \text{div } \overline{\text{grad } \vec{u}}$$

Exercise 5.10. Formulate the Navier-Stokes equation for stationary flow of incompressible liquid.

Exercise 5.11. Motivate the origin and assumptions of Euler's equation for inviscid gas

$$(5.33b) \quad \rho \frac{d\vec{u}}{dt} = - \text{grad } p$$

Exercise 5.12. Motivate the origin and assumptions of the equation of motion for the inviscid and neutral ($\rho_e = 0$) plasma in the magnetohydrodynamic flow:

$$(5.34) \quad \rho \frac{d\vec{u}}{dt} = \frac{1}{c} \vec{i} \times \vec{B} - \text{grad } p$$

Exercise 5.13. A rigid body (e.g. an aircraft or submarine) is immersed in the fluid and moves rectilinearly with a constant velocity \vec{w} . We erect an imaginary referential region inside and outside of the body. Assuming the region to be rigid and stiffly connected with the body, we find that the referential velocity field \vec{w} is stationary and homogenous: $\partial \vec{w} / \partial t = 0$, $\overline{\text{grad } \vec{w}} = 0$, $\text{div } \vec{w} = 0$. The fluid flows round the body in accordance with the absolute velocity field \vec{u} and relative velocity field $\vec{u}' = \vec{u} - \vec{w}$. Prove that within the fluid region the following equation is valid:

$$(5.35) \quad \frac{d(\rho \vec{u}')}{dt} = \vec{F} - \text{div}(\vec{\Pi} + \overline{\rho \vec{u}' \vec{u}'})$$

Compare this equation with the momentum balance for immobile point (5.30b), showing the correspondences: $\vec{u}' \leftrightarrow \vec{u}$, $\frac{d}{dt} \rightarrow \frac{\partial}{\partial t}$ at $\vec{w} \leftrightarrow 0$. Explain whether Eq.(5.35) keeps its validity in case of an accelerated or rotary motion of the body.

Instructions. Substitute $\vec{u} = \vec{u}' + \vec{w}$ into the balance (5.31b), use the rule (1.31b) and Eq.(5.2b), and take into consideration

the properties of field \bar{w} .

Additional exercises. The correspondence between Eqs.(5.35) and (5.31b) makes possible to calculate the time-process of the single flow perturbation at substantial or immobile point caused by a rigid body passing in an uniform (rectilinear) motion. If the relative flow round the body with the velocity \bar{u}' is stationary with respect to referential system $V_{\bar{w}}$, then the left-hand side of Eq.(5.35) vanishes. Prove that in such case we have

$$(5.36a) \quad \frac{\partial(\varrho \bar{u})}{\partial t} + \overline{\text{div}}(\varrho \bar{u}' \bar{u}) = \overline{\text{div}}(\varrho \bar{u}' \bar{u}')$$

$$(5.36b) \quad \varrho \frac{d\bar{u}}{dt} \rightarrow \bar{u} = \overline{\text{div}}(\varrho \bar{u}' \bar{u}')$$

(instruction: transform the substantial Reynoldsian).

If we know the fields ϱ and \bar{u}' of the stationary relative flow (the right-hand side), then we are able to determine the non-stationary field of absolute acceleration of the fluid $\frac{d\bar{u}}{dt} \rightarrow \bar{u}$. Prove that in an incompressible fluid

$$(5.37) \quad \frac{\partial \bar{u}}{\partial t} = - \overline{\bar{w}' \text{grad } \bar{u}}$$

Exercise 5.14. Formulate the balance of momentum and the equation of motion for a rocket at the assumptions given in Ex. 5.1 (Fig. 5.1). In order to simplify the problem, assume that the rocket is moving along an approximately straight trajectory through a medium rarefied to such an extent that on the outer surface of the hull s_s there is no stress ($\bar{\Pi} = 0$), and on the outlet surface s_n the gas pressure p is negligible as compared to $\varrho_g(\bar{u}_g^2)$.

Instructions. The region $V_{\bar{w}}$ consists of the solid part moving with velocity $\bar{u} = \bar{w}$, and gas-filled part, where the velocity of substance $\bar{u} \neq \bar{w}$. Assume that the mass-share of the gaseous part is negligibly small, and use the kinematic property of the solid part (see Ex. 5.1) in order to shift the homogenous vector \bar{w} before the integral sign in the expression for momentum stored within $V_{\bar{w}}$.

Make appropriate substitutions into balance (5.31a) and take into consideration that $\bar{u} = \bar{w}$ on the hull surface s_s , and $\bar{u} = \bar{u}_g$, $\bar{u} - \bar{w} = \bar{u}_g$ on the surface s_n . Use the rule (1.20a) and Eq.(5.6).

The results:

$$(5.38a) \quad \frac{d(m\bar{w})}{dt} \rightarrow \bar{w} = m\bar{g} + \bar{u}_g \frac{dm}{dt} \rightarrow \bar{w} \quad \text{balance of momentum}$$

$$(5.38b) \quad m \frac{d\bar{w}}{dt} \rightarrow \bar{w} = m\bar{g} + \bar{u}_g \frac{dm}{dt} \rightarrow \bar{w} \quad \text{equation of motion}$$

Is the thrust $\bar{u}_g \frac{dm}{dt} \rightarrow \bar{w}$ a "true" force in the meaning of the Second Law of Motion?

5.9. The Work

As a scalar product of force and substantial displacement $d\bar{l} = \bar{u} dt$, the work can be expressed as follows, with use of the rule (1.21):

$$(5.39a) \quad \left. \begin{array}{l} \text{elementary work} \\ \text{of body force} \end{array} \right\} = (\bar{F} d^3V d\bar{l}) = (\bar{F} \bar{u}) d^3V dt$$

$$(5.39b) \quad \left. \begin{array}{l} \text{elementary work} \\ \text{of surface force} \end{array} \right\} = - (d^2\bar{s} \cdot \bar{n} d\bar{l}) = - (\bar{n} \cdot \bar{u} d^2\bar{s}) dt$$

The work of all the forces exerted on a substantial balance region per unit time is called the power:

$$(5.40a) \quad \text{power} = \frac{\text{work}}{\text{time}} = \iiint_{V_{\bar{u}}} (\bar{F} \bar{u}) d^3V - \oint_{S_{\bar{u}}} (\bar{n} \cdot \bar{u} d^2\bar{s})$$

Applying the GGO theorem (1.34a) we may obtain an entirely volume integral of the power density (power per unit volume). Using the rule (1.30b) we transform the power density in order to classify various kinds of work:

$$(5.40b) \quad \left. \begin{array}{l} \text{power} \\ \text{density} \end{array} \right\} = \underbrace{(\bar{F} \bar{u})}_{\text{corresponding to}} - \underbrace{\text{div } \bar{n} \cdot \bar{u}}_{\text{"total work"}} = \underbrace{(\bar{F} \bar{u}) - (\bar{u} \text{ div } \bar{n})}_{\text{"mechanical work"}} - \underbrace{(\bar{n} \bar{g} \bar{r} \bar{a} \bar{d} \bar{u})}_{\text{"thermodynamic work"}}$$

The "total work" consists of "mechanical work" resulting from the translation of a body (direct dependence on \bar{u}) and "thermo-

dynamic work" accompanying the deformation of a body (dependence on $\bar{g}\bar{r}\bar{a}\bar{d}\bar{u}$). It will appear in Eq.(5.41) that only the "mechanical work" is equal to the change of storage of kinetic energy, while the "thermodynamic work" is expressing conversion between kinetic energy and internal energy.

In some textbooks of mechanics the work has been defined as an equivalent of energy expended on the displacement of a body by action of forces. This definition proved to be vague because neither the kind of energy involved has been specified (a crucial question in the light of the energy conservation principle), nor the "expenditure" explained (conversion or transfer?). In the language of thermodynamics, the work is generally restricted to its "thermodynamic" part (conversion, in fact), usually with the effects of viscosity excluded. It is familiar to use the term "electric work" for the conversion of electromagnetic energy with the exception of the hysteresis effects.

Having a vague meaning and being not clearly associated with the terms of storage, production and transfer of energy, the concept of work seems to be of a dubious interpretative value in the balance theory. Since the use of conversions between specific energies and of specified transfers between the system and its surroundings makes a very consistent and comprehensible image of the energetic processes, we may dispense with the concept of work in our further exposition, with no harm to all the reasoning.

Exercise 5.15. Express the mass-density of elementary "thermodynamic work" for inviscid gas ($\bar{\Pi} = \bar{\Pi}_p = \bar{I}_p$).

Instructions. Apply the rule (1.29a) and balance (5.4c) to the proper term of Eq.(5.40b). Notice that Eq.(5.40b) describes the volume-density of power, and not the mass-density of work. The result: $-p \frac{dv}{\rightarrow v}$ (in the thermodynamics, the substantial increment of v is simply denoted by dv).

5.10. The Balance of Kinetic Energy

Being the product of mass and the one-half of squared velocity, the kinetic energy is a scalar SQ. According to definitions (5.7) and (5.9a), the kinetic energy of continuous substance has the storage mass-density $(\bar{u}^2/2)$, and the volume-density $\varrho(\bar{u}^2/2)$. To be strict, the above densities characterize only the translational kinetic energy in the phenomenological meaning, i.e. the energy of the ordered translatory motion of grains of substance, connected with non-zero macroscopic momentum. All those provisions exclude the energies due to the diffusive and the disordered thermal motions, which are included in the internal energy.

In the polar fluid and the Cosserat elastic medium, the phenomenological kinetic energy contains in addition the energy of intrinsic rotation, i.e. of an ordered rotary motion of grains which are spinning around their own axes. The mass-density of this additional energy is $I(\bar{\omega}^2/2)$, where I denotes the geometric moment of inertia of each of identical grains, and $\bar{\omega}$ is the macroscopically averaged angular velocity of grains. The further exposition, however, will be confined to the ordinary substance, with no ordered intrinsic rotation.

Multiplying the balance of momentum (5.29b) by \bar{u} in a scalar product

$$(5.41) \quad (\varrho \frac{d\bar{u}}{dt} \rightarrow \bar{u} \bar{u}) = \varrho \frac{d}{dt} \rightarrow \bar{u} (\frac{\bar{u}^2}{2}) = (\bar{F} \bar{u}) - (\bar{u} \operatorname{div} \bar{\Pi})$$

and using the rule (1.30b), we obtain an equation of the type of Eq.(5.19a), so we conclude that this is the balance of kinetic energy at a substantial point:

$$(5.42a) \quad \varrho \frac{d}{dt} \rightarrow \bar{u} (\frac{\bar{u}^2}{2}) = (\bar{F} \bar{u}) + (\bar{\Pi} \bar{g} \bar{r} \bar{a} \bar{d} \bar{u}) - \operatorname{div} \bar{\Pi} \bar{u}$$

Applying the theorem on equivalence of balances (5.19a) and (5.21a), we obtain the balance of kinetic energy at an immobile point:

$$(5.42b) \quad \frac{\partial}{\partial t} \left\{ \rho \left(\frac{\bar{u}^2}{2} \right) \right\} = (\bar{F} \bar{u}) + (\bar{\Pi} \bar{g} \bar{r} \bar{a} \bar{d} \bar{u}) - \text{div} \left\{ \bar{\Pi} \bar{u} + \rho \left(\frac{\bar{u}^2}{2} \right) \bar{u} \right\}$$

To be strict, the balances (5.42) deal with the translational kinetic energy only. In the polar fluid or the Cosserat elastic medium, we have to add the balance of kinetic energy of intrinsic rotation, derived from the scalar multiplication of the balance of intrinsic angular momentum by the angular velocity $\bar{\omega}$ of the ordered rotation.

Replacing \bar{F} by substitution of Eq.(5.22) into the balances (5.42) we are able to reveal the specific conversions between the kinetic energy and other energies:

$$(5.43a) \quad \left. \begin{array}{l} \text{density of production} \\ \text{of kinetic energy} \\ \text{at expense/for benefit} \\ \text{of energy:} \end{array} \right\} = \underbrace{(\bar{F}_{ed} \bar{u})}_{\text{electro- magnetic}} + \underbrace{(\bar{F}_g \bar{u})}_{\text{gravi- tational}} + \underbrace{(\bar{\Pi} \bar{g} \bar{r} \bar{a} \bar{d} \bar{u})}_{\text{internal}}$$

The ascription of particular terms to specific energies is based on the following reasoning. Since the term $(\bar{F}_{ed} \bar{u})$ with the negative sign appears in the balance of electromagnetic energy (4.49), it proves that in the balance of the sum of kinetic and electromagnetic energies their mutual conversion vanishes. In the same way, the term $(\bar{F}_g \bar{u})$ with the negative sign will appear in the balance of gravitational energy (5.44a). In consideration of the arbitrariness of the angle between vectors \bar{F}_{ed} or \bar{F}_g and \bar{u} , both conversions are of reversible character: the kinetic energy may be either created or annihilated at the expense or for the benefit of electromagnetic and gravitational energies.

The ascription of the term $(\bar{\Pi} \bar{g} \bar{r} \bar{a} \bar{d} \bar{u})$ to conversion between kinetic energy and internal energy results from the principle of conservation of the total energy, which will be discussed in a subsequent section (see Eq.5.56). This conversion is partly reversible and irreversible, the latter being the annihilation of kinetic energy due to viscous dissipation.

It results from the balance (5.42b) and classification (5.13a) that

$$(5.43b) \left. \begin{array}{l} \text{transfer-flux density} \\ \text{of kinetic energy} \\ \text{through immobile surface} \end{array} \right\} = \underbrace{\bar{\bar{\Pi}}^* \bar{u}}_{\text{non-convective}} + \underbrace{\varrho \left(\frac{\bar{u}^2}{2} \right) \bar{u}}_{\text{convective}}$$

kind of transfer: non-convective convective

The non-convective transfer is connected with the field $\bar{\bar{\Pi}}$, and therefore is called the stress (or mechanical) transfer of energy. The kinetic energy flows in the direction of vector $\bar{\bar{\Pi}}^* \bar{u}$, which may be different from the direction of substance's velocity \bar{u} . For example the shear viscosity phenomenon causes that the transport of kinetic energy is perpendicular to the transport of mass.

In the inviscid fluid, where the stress is of purely pressure character, $\bar{\bar{\Pi}} = \bar{\bar{\Pi}}_p = \bar{\Pi}_p$, $\bar{\bar{\Pi}}^* \bar{u} = p \bar{u}$, the directions of flows of mass and kinetic energy are identical. From this we must not draw an erroneous conclusion that the transfer characterized by the density $p \bar{u}$ is of convective character. While the mass cannot cross a substantial surface (Eq.5.1a), the kinetic energy flows through the same surface with the transfer-flux density $p \bar{u}$.

5.11. The Balances of Gravitational Energy and Mechanical Energy

The virtual site of the gravitational energy, as well as of the electromagnetic energy, is the field in space, either filled with substance or empty. In the terrestrial conditions, however, we can take a fictitious but practical assumption that the gravitational energy is stored within substance and can be carried by substance. In such case the storage mass-density of gravitational energy, as a SQ, is simply equal to the excessive gravitational potential $\Gamma > 0$, and the volume-density is $\varrho \Gamma$.

At the assumption given, the balance of gravitational energy at a substantial point results from multiplication of relation (5.25b) by ϱ with the substitution of Eq.(5.23a), and the balance at an

immobile point is obtained from the equivalence of basic balances (5.19a) and (5.21a):

$$(5.44a) \quad \rho \frac{d}{dt} \bar{u} = - (\bar{F}_g \bar{u}) = - \rho (\bar{g} \bar{u})$$

$$(5.44b) \quad \frac{d}{dt} (\rho \Gamma) = - (\bar{F}_g \bar{u}) - \text{div}(\rho \Gamma \bar{u})$$

From the comparison of Eqs.(5.44) and (5.43a) we conclude that the production of gravitational energy is nothing more than the reversible conversion with kinetic energy. The absence of divergence term in the balance (5.44a) proves that, in the terrestrial conditions, the non-convective transfer of gravitational energy does not occur at all, though the convective transfer does exist (balance 5.44b). Strictly speaking, the discussed energy, being a field energy, is subjected to a non-convective transfer called the gravitational radiation (similar to electromagnetic radiation). This phenomenon, however, does not play any role in the terrestrial conditions.

Since the kinetic energy and the gravitational energy in terrestrial conditions are both of substantial character, sometimes it is convenient to add them up, the sum obtained being called the mechanical energy. Its balance at a substantial point results from association of Eqs.(5.44a), (5.43a), and (5.42a):

$$(5.45) \quad \rho \frac{d}{dt} \bar{u} \left\{ \left(\frac{\bar{u}^2}{2} \right) + \Gamma \right\} = (\bar{F}_{ed} \bar{u}) + (\bar{\Pi} \bar{g} \bar{r} \bar{a} \bar{d} \bar{u}) - \text{div} \bar{\Pi}' \bar{u}$$

The production of mechanical energy is due to conversions with the electromagnetic and internal energies. Comparing Eqs.(5.45) and (5.42a), we conclude that the non-convective transfers of mechanical energy and kinetic energy are identical.

5.12. The Balance of Elastic Energy at Small Strain

From the microphysical viewpoint, the elastic energy is the energy of structural bonds of the body. From the phenomenological

viewpoint, it forms a part of the internal energy, and so it is a SQ of certain mass-density ϵ_e . According to the classical theory of elasticity, the volume-density of the elastic energy, for a perfectly elastic body at small strain, is as follows:

$$(5.46) \quad \rho \epsilon_e = \mu_b \frac{\text{div}^2 \delta \bar{r}}{2} + \mu_s \frac{(\bar{g} \bar{r} \bar{a} \bar{d} \text{ds}^2 \delta \bar{r})}{2} \geq 0$$

In case of the Cosserat elastic medium, we have to add some quadratic forms connected with antisymmetric stress and couple-stress (resulting from torsional strain of body's microstructure). Those effects do not occur in the ordinary elastic medium discussed below.

The elastic energy is a non-negative excessive energy above the level corresponding to the undistorted state ($\bar{g} \bar{r} \bar{a} \bar{d}^S \delta \bar{r} = 0$). It is worth to emphasize that at large values of coefficients μ_b and μ_s a small strain may cause considerable stress (5.26a), and enormous density of elastic energy (5.46).

As a constituent of internal energy, the elastic energy is being subjected to conversion with kinetic energy in the course of deformation, at which the velocity of substance \bar{u} must differ, at least locally, from zero. Consequently, the term $(\bar{\Pi} \bar{g} \bar{r} \bar{a} \bar{d} \bar{u})$ of Eq.(5.43a), which in case of a perfectly elastic body ($\bar{\Pi} = \bar{\Pi}_e$) is equal to $(\bar{\Pi}_e \bar{g} \bar{r} \bar{a} \bar{d} \bar{u})$, expresses the conversion between kinetic energy and elastic energy.

It is possible to show that at small strain we have

$$(5.47a) \quad (\bar{\Pi}_e \bar{g} \bar{r} \bar{a} \bar{d} \bar{u}) \cong - \frac{d}{dt} \bar{u} \left\{ \mu_b \frac{\text{div}^2 \delta \bar{r}}{2} + \mu_s \frac{(\bar{g} \bar{r} \bar{a} \bar{d} \text{ds}^2 \delta \bar{r})}{2} \right\}$$

where $\bar{\Pi}_e$ is defined by Hooke's law (5.26a) (the proof has been omitted as very tedious and requiring intricate transformations of the tensor calculus). Combining Eqs.(5.46) and (5.47a), we obtain:

$$(5.47b) \quad (\bar{\Pi}_e \bar{g} \bar{r} \bar{a} \bar{d} \bar{u}) \cong - \frac{d(\rho \epsilon_e)}{dt} \bar{u}$$

Since at small strain the mass density remains nearly unchanged along a substantial displacement (there exists a proof that the error involved is negligible), we can shift ϱ before the differential sign to obtain an equation of the type of Eq.(5.19a). This will be the balance of elastic energy at a substantial point:

$$(5.48) \quad \varrho \frac{d\varepsilon_e}{dt} \rightarrow \bar{u} = - (\bar{\Pi}_e \bar{g} \bar{r} \bar{a} \bar{d} \bar{u})$$

The absence of divergence term proves that the elastic energy cannot be transferred in the non-convective way. The change of elastic energy stored within a substantial region may be solely due to conversion with the kinetic energy. From the structure of Eq. (5.47a) we conclude that the conversion is of the reversible character: $(\bar{\Pi}_e \bar{g} \bar{r} \bar{a} \bar{d} \bar{u}) \gtrless 0$. It is worth to notice that in spite of analogous structure of laws of Hooke (5.26a) and Newton (5.26b), the viscous dissipation of kinetic energy for the benefit of internal energy is an irreversible conversion: $(\bar{\Pi}_v \bar{g} \bar{r} \bar{a} \bar{d} \bar{u}) \leq 0$ (see Sec.5.14).

5.13. The Energy Balance of a Perfectly Elastic Body at Small Strain

Consider a perfectly elastic body, being deformable to a small extent only, and not subjected to any electromagnetic exertion. Substituting $\bar{\Pi} = \bar{\Pi}_e$ and $\bar{F}_{ed} = 0$ into Eq.(5.45) and using Eq.(5.48), we obtain the balance of the sum of mechanical energy and elastic energy at a substantial point of the described body:

$$(5.49) \quad \varrho \frac{d}{dt} \rightarrow \bar{u} \left\{ \frac{\bar{u}^2}{2} + \bar{\Gamma} + \varepsilon_e \right\} = - \operatorname{div} \bar{\Pi}_e' \bar{u}$$

The absence of production term proves that the sum of the energies mentioned is a conservative SQ, so it can be neither created nor annihilated. For example when the body is falling, there is a conversion between the gravitational energy and kinetic energy, and when the body hits an obstacle keeping its perfect elasticity, the conversion occurs between the kinetic energy and elastic energy.

In the assumed conditions, the only non-convective transfer of energy is due to the elastic stress. The elastic transfer-flux density $\overline{\overline{\Pi}}_e \overline{\overline{u}}$ is called Umov's vector. It has to be emphasized that this kind of transfer concerns the kinetic energy, and not the elastic energy. Umov's vector makes it possible to transmit the kinetic energy by means of an elastic wave, and to deliver the power by means of translatory and rotary motion of mechanisms.

The elastically-transferred power is the non-convective transfer of kinetic energy through an imaginary substantial surface within an elastic body (e.g. through a cross-section of mechanical element)

$$(5.50a) \left. \begin{array}{c} \text{elastically-} \\ \text{-transferred} \\ \text{power} \end{array} \right\} = \iint_{\vec{s}_{\vec{u}}} (d^2\vec{s} \cdot \overline{\overline{\Pi}}_e \overline{\overline{u}}) = \iint_{\vec{s}_{\vec{u}}} (\overline{\overline{d^2\vec{s} \cdot \overline{\overline{\Pi}}_e}} \cdot \overline{\overline{u}})$$

(the equivalence of both integrals results from the rule 1.21).

Fig. 5.3

In case of a push-pull rod (Fig. 5.3), its motion is of purely translatory character. Consequently, the velocity \overline{u} remains constant over the whole cross-section $\vec{s}_{\vec{u}}$, and so it may be shifted before the integral sign:

$$(5.50b) \left. \begin{array}{c} \text{power transferred} \\ \text{along push-pull rod} \end{array} \right\} = (\overline{u} \underbrace{\iint_{\vec{s}_{\vec{u}}} \overline{\overline{d^2\vec{s} \cdot \overline{\overline{\Pi}}_e}}}_{\substack{\text{force on} \\ \text{cross-section}}})$$

In this case, the transferred power is the product of velocity and force occurring on the cross-section and due to compressive or tensile stress.

Fig. 5.4.

In case of a drive-shaft (Fig. 5.4), its motion is of purely rotary character. Consequently, the velocity \overline{u} depends on the angular velocity $\overline{\omega}_0$ and the arm of rotation \overline{r} : $\overline{u} = \overline{\omega}_0 \times \overline{r}$. Since $\overline{\omega}_0$ remains constant over the cross-section of the shaft $\vec{s}_{\vec{u}}$, it may be shifted before the integral sign in Eq.(5.50a), the transformation (1.19a) being previously applied:

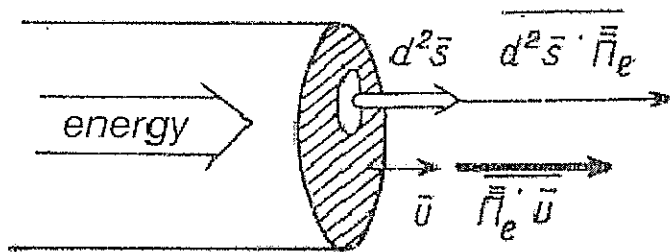


Fig. 5.3. Non-convective flow of kinetic energy through the cross section of a push-pull rod

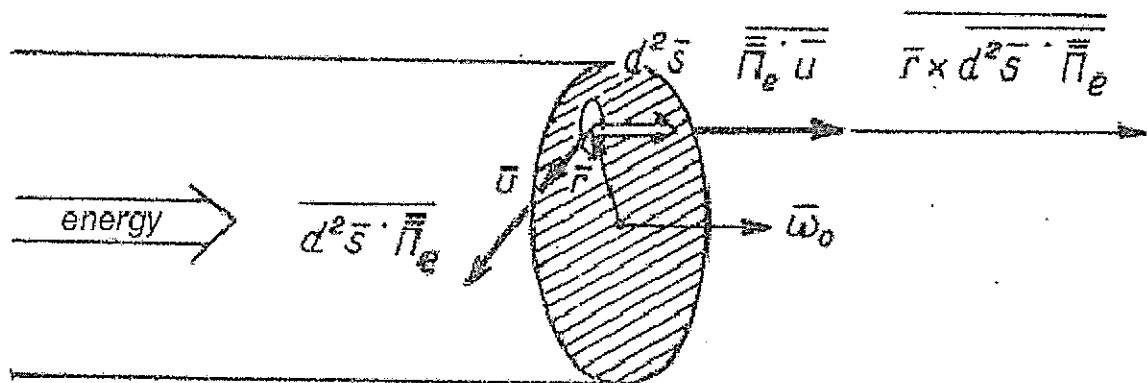


Fig. 5.4. Non-convective flow of kinetic energy through the cross-section of a drive shaft

$$(5.50c) \left. \begin{array}{l} \text{power transferred} \\ \text{along drive-shaft} \end{array} \right\} = (\bar{\omega}_0 \underbrace{\int_{\vec{s}_{\vec{u}}} \vec{r} \times d^2\vec{s} \cdot \vec{\bar{\Pi}}_e}_{\text{torque on cross-section}})$$

This time the transferred power is the product of angular velocity and the torque occurring on the cross-section and due to torsional stress.

Since the cross-section $\vec{s}_{\vec{u}}$ is an open surface, its orientation may be arbitrary. If it appears that the expressions (5.50b,c) are positive scalars, the kinetic energy flows along the rod or shaft towards the orientation of $\vec{s}_{\vec{u}}$, and if negative, it flows oppositely.

It is difficult to explain the described processes with use of the concept of work. Since in a stationary motion both integrals in Eq.(5.40a) are incessantly equal to zero, the elastic element performs no work at all. We might try to use the idea of "transfer of work" but we do not need such an explanation, having already a very suggestive transport interpretation: the discussed process is the non-convective flow of kinetic energy, ceasing to exist when the velocity of mechanism vanishes.

5.14. The Balance of Kinetic Energy of the Fluid

The general balance of kinetic energy (5.42) can be re-written in a more specified form for the fluid. According to Eq.(4.27c), we split-up the resultant stress $\vec{\bar{\Pi}}$ into the pressure and viscous stresses, $\vec{\bar{\Pi}}_p$ and $\vec{\bar{\Pi}}_v$, respectively. Then we substitute the decomposed stress into the terms of conversion (with internal energy) and non-convective transfer, $(\vec{\bar{\Pi}} \vec{\bar{g}} \vec{r} \vec{a} \vec{d} \vec{u})$ and $\text{div } \vec{\bar{\Pi}} \vec{u}$, respectively.

It is easy to prove (Ex. 5.15) with the help of Eqs.(5.27a), (1.29a), and (5.4c) that

$$(5.51a) \quad (\vec{\bar{\Pi}}_p \vec{\bar{g}} \vec{r} \vec{a} \vec{d} \vec{u}) = \varphi_p \frac{dv}{dt} \vec{u} \gtrless 0$$

The above expression describes that part of conversion which is reversible and can occur in a compressible fluid only: at the expansion process ($\frac{dv}{dt} \vec{u} > 0$), the kinetic energy is being created at the expense of internal energy (the gas within an adiabatic system is cooling down), and at the compression process ($\frac{dv}{dt} \vec{u} < 0$), the kinetic energy is being annihilated for the benefit of internal energy (the gas is warming up).

It is also possible to prove (Ex. 5.16) that for the fluid obeying Newton's law of viscosity (5.26b), we have

$$(5.51b) \quad (\bar{\bar{\Pi}}_V \bar{\bar{g}} \bar{\bar{r}} \bar{\bar{a}} \bar{\bar{d}} \bar{u}) = - \left\{ \gamma_b \operatorname{div}^2 \bar{u} + \gamma_s (\bar{\bar{g}} \bar{\bar{r}} \bar{\bar{a}} \bar{\bar{d}} s^2 \bar{u}) \right\} \leq 0$$

This part of conversion, as a negative quadratic form, is the irreversible annihilation of kinetic energy for the benefit of internal energy. Since it is due to the viscosity phenomenon, it is called the viscous dissipation. The created internal energy assumes the form of "frictional heat".

It results from the Second Law of Thermodynamics that the left-hand side expression (5.51b) is non-positive-valued for any kind of substance, including those which are not obeying the law (5.26b). Consequently, we are allowed to formulate the general classifying decomposition of conversion between the kinetic and internal energies of the fluid:

$$(5.52a) \quad (\bar{\bar{\Pi}} \bar{\bar{g}} \bar{\bar{r}} \bar{\bar{a}} \bar{\bar{d}} \bar{u}) = \underbrace{\rho p \frac{dv}{dt} \vec{u}}_{\geq 0} + \underbrace{(\bar{\bar{\Pi}}_V \bar{\bar{g}} \bar{\bar{r}} \bar{\bar{a}} \bar{\bar{d}} \bar{u})}_{< 0}$$

Using Eqs.(4.27c) and (1.18d) we obtain another classifying decomposition, this time concerning the transfer:

$$(5.52b) \quad \underbrace{\bar{\bar{\Pi}} \bar{\bar{u}}}_{\substack{\text{non-convective} \\ \text{transfer-flux density} \\ \text{of fluid's kinetic energy}}} = \underbrace{p \bar{u}}_{\substack{\text{by means of} \\ \text{pressure}}} + \underbrace{\bar{\bar{\Pi}}_V \bar{\bar{u}}}_{\substack{\text{by means of} \\ \text{viscosity}}}$$

As it was stated before (Sec. 5.10), the transfer by means of pressure has the same direction as the velocity \bar{u} , while the transfer by means of viscosity is, in general, directed differently. Only this part of transfer which is due to the bulk viscosity (spherical tensor in Eq. 5.26b) keeps parallelism with \bar{u} , at the sense unchanged (compression) or opposite (expansion). In the flat flow of incompressible fluid, the shear viscosity phenomenon induces the kinetic energy to flow perpendicularly to the streamlines, from the layer of greater velocity to the layer of smaller velocity (Fig. 5.5a).

Fig. 5.5

It is due to vector $p\bar{u}$ that the kinetic energy can be non-convectively transmitted by means of the pressure field of acoustic waves. The same kind of transfer is the principle of functioning of hydraulic and pneumatic piston machines (pumps, compressors, engines) transferring the kinetic energy between the working fluid and the piston.

Consider a piston moving along the cylinder with velocity \bar{u} (Fig. 5.5b). The substantial interface $\underline{s}_{\bar{u}}$ separating the piston from the fluid-filled interior of the cylinder consists of surface elements $d^2\bar{s}$. During time dt each of them sweeps a volume element of the 3rd infinitesimality order $d^3V = (\bar{u} dt d^2\bar{s})$. During the same lapse of time, the finite surface $\underline{s}_{\bar{u}}$ sweeps a spatial slice of volume of the 1st infinitesimality order (being in fact a positive or negative increment of volume) equal to

$$(5.53a) \quad \iint_{\underline{s}_{\bar{u}}} (\bar{u} dt d^2\bar{s}) = \iint_{\underline{s}_{\bar{u}}} d^3V = dV \gtrless 0$$

If the pressure p is identical at each point of the surface $\underline{s}_{\bar{u}}$, which is oriented towards the interior of the piston, then the energy transferred through $\underline{s}_{\bar{u}}$ during dt is:

$$(5.53b) \quad \left. \begin{array}{l} \text{kinetic energy} \\ \text{transferred} \\ \text{non-convectively} \\ \text{from fluid to piston} \end{array} \right\} = \iint_{\underline{s}_{\bar{u}}} (p\bar{u} d^2\bar{s}) dt = p dV \gtrless 0$$

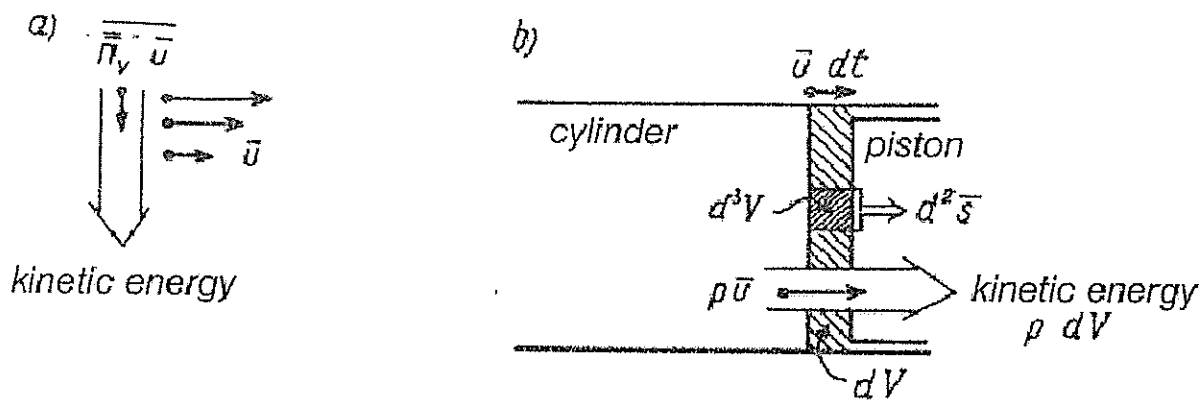


Fig. 5.5. Non-convective exchange of kinetic energy:
 (a) in flow of a viscous non-compressible fluid
 (b) during movement of a piston in cylinder

It is worth to emphasize once again the non-convective character of transfer: the substance cannot penetrate into the piston, while the kinetic energy flows from the fluid to the piston (when $dV > 0$), or vice versa (when $dV < 0$).

Having discussed some aspects of conversion and transfer, we go back to the balance itself. We shall present only two most useful forms of the balance of fluid's kinetic energy, obtained from substitution of Eqs.(5.52a,b) into Eqs.(5.42):

$$(5.54a) \quad \rho \frac{d}{dt} \int_V \frac{\bar{u}^2}{2} = (\bar{F} \bar{u}) + \rho p \frac{dv}{dt} \bar{u} + (\bar{\Pi}_V \bar{g} \bar{r} \bar{a} \bar{d} \bar{u}) - \text{div} \{ p \bar{u} + \bar{\Pi}_V \bar{u} \}$$

$$(5.54b)$$

$$\frac{\partial}{\partial t} \left\{ \rho \left(\frac{\bar{u}^2}{2} \right) \right\} = (\bar{F} \bar{u}) + \rho p \frac{dv}{dt} \bar{u} + (\bar{\Pi}_V \bar{g} \bar{r} \bar{a} \bar{d} \bar{u}) - \text{div} \left\{ p \bar{u} + \bar{\Pi}_V \bar{u} + \rho \left(\frac{\bar{u}^2}{2} \right) \bar{u} \right\}$$

Exercise 5.16. Derive Eq.(5.51b).

Instruction. Apply the rules (1.29a) and (1.17f).

Exercise 5.17. Derive the hydraulic equation of Bernoulli:

$$(5.55) \quad p/\rho + u^2/2 + gh = \text{const} \quad \text{along the streamline}$$

Validity of this equation is limited to the stationary flow of incompressible and inviscid liquid, in the absence of any electromagnetic exertion.

Instructions. It is convenient to start with the mechanical energy balance at immobile point, obtained from Eq.(5.45) with the help of the theorem on equivalence of balances (5.19a) and (5.21a). It results from the limitations of the Bernoulli equation (Eq.5.5b, $\bar{\Pi}_V, \bar{F}_{ed} = 0$) that in this specific case the mechanical energy is a conservative quantity, and at a stationary flow ($\partial/\partial t = 0$) its balance resolves itself into equation $\text{div} \dots = 0$. The div sign is followed by the product of vector \bar{u} and a certain sum of scalars, to which we apply Eq.(5.24) and denotation $(\bar{u}^2) = u^2$. Transforming the divergence term with use of the rule (1.30a) and applying the incompressibility condition (5.5b) once again, we obtain the equat-

ion: $(\bar{u} \overline{\text{grad}} \dots) = 0$. Since $\bar{u} \neq 0$ (generally), so at any place the projection of the vector $\overline{\text{grad}} \dots$ on the direction tangent to the streamline must be equal to zero. Dividing the gradiented function by $Q = \text{const}$, we obtain the final result.

5.15. The Balance of Internal Energy

In the phenomenological approach, the internal energy is regarded as a set of all those specific energies that are connected neither with the macroscopic motion of substance nor with the long-range force fields (Sec. 2.7). Virtually, this is the energy of those motions of grains that have mutually compensated momentum (the thermal energy, Eqs. 2.7c and 2.8b), as well as of the short-range fields of the grain interaction which vanish when averaged macroscopically (the bond energy, compare Eqs. 2.13b, 2.14b). The elastic energy the energy of intra-atomic bonds, Sec. 5.12), for example, is also a constituent of the internal energy.

Since the grains are the carriers of the thermal motion energy and the practical range of close interaction does not much exceed the grain dimensions, the internal energy may be regarded as a scalar SQ of the storage mass-density ϵ_I and the volume-density $\rho \epsilon_I$ (see Eqs. 5.7 and 5.9a).

From the phenomenological viewpoint, the sum of external energy and internal energy gives the total energy (see Table 2.15), which is a conservative quantity characterized by the lack of production (Eq. 4.32). Consequently, the production of internal energy must be equal to the minus production of external energy, the latter consisting of electromagnetic, kinetic, and gravitational energies (Table 2.15). Selecting the production terms from Eqs. (4.49), (5.43a) and (5.44b), we conclude that

(5.56) the production (volume-) density of internal energy =

$$\begin{aligned}
 &= G_{pm} + (\overline{E}_{\vec{u} \rightarrow \vec{u}}) + (\overline{F}_{ed} \vec{u}) - (\overline{F}_{ed} \vec{u}) - (\overline{F}_g \vec{u}) - (\overline{\Pi} \overline{g} \overline{r} \overline{a} \overline{d} \vec{u}) + (\overline{F}_g \vec{u}) = \\
 &= \underbrace{G_{pm} + (\overline{E}_{\vec{u} \rightarrow \vec{u}})}_{\text{conversion with electromagnetic energy}} - \underbrace{(\overline{\Pi} \overline{g} \overline{r} \overline{a} \overline{d} \vec{u})}_{\text{conversion with kinetic energy}}
 \end{aligned}$$

According to classification (5.13a), the internal energy, as a SQ, is subjected to the convective transfer of flux density $\varrho \varepsilon_I \vec{u}$ and the non-convective one, the flux density of which will be denoted by \vec{J}_{nc} .

Now, in the basic balances (5.19a) and (5.21a), we replace ε with ε_I , $\varrho \mu$ with the expression (5.56), and $\vec{H}_{\vec{u}}$ with \vec{J}_{nc} , in order to obtain the balances of internal energy at substantial and immobile points:

$$(5.57a) \quad \varrho \frac{d\varepsilon_I}{dt} \vec{u} = G_{pm} + (\overline{E}_{\vec{u} \rightarrow \vec{u}}) - (\overline{\Pi} \overline{g} \overline{r} \overline{a} \overline{d} \vec{u}) - \text{div } \vec{J}_{nc}$$

$$(5.57b) \quad \frac{\partial(\varrho \varepsilon_I)}{\partial t} = G_{pm} + (\overline{E}_{\vec{u} \rightarrow \vec{u}}) - (\overline{\Pi} \overline{g} \overline{r} \overline{a} \overline{d} \vec{u}) - \text{div}(\vec{J}_{nc} + \varrho \varepsilon_I \vec{u})$$

The substitution (5.52a) makes the above balances specified for the case of fluid:

$$(5.58a) \quad \varrho \frac{d\varepsilon_I}{dt} \vec{u} = G_{pm} + (\overline{E}_{\vec{u} \rightarrow \vec{u}}) - \varrho p \frac{dv}{dt} \vec{u} - (\overline{\Pi}_{\vec{v}} \overline{g} \overline{r} \overline{a} \overline{d} \vec{u}) - \text{div } \vec{J}_{nc}$$

$$(5.58b) \quad \frac{\partial(\varrho \varepsilon_I)}{\partial t} = G_{pm} + (\overline{E}_{\vec{u} \rightarrow \vec{u}}) - \varrho p \frac{dv}{dt} \vec{u} - (\overline{\Pi}_{\vec{v}} \overline{g} \overline{r} \overline{a} \overline{d} \vec{u}) - \text{div}(\vec{J}_{nc} + \varrho \varepsilon_I \vec{u})$$

Recapitulating considerations given in Secs. 4.14, 4.13, 5.12 and 5.14 (Eqs. 4.46, 4.40, 5.48, 5.52a), we are allowed to assume the following interpretations of expressions for the internal energy conversion concerning specific phenomena and media:

- (5.59a) $-(\bar{K} \bar{I}_{\rightarrow \bar{u}}) \geq 0$ { reversible conversion of bond energy (electrochemical phenomena) or thermal energy (thermoelectric phenomena) with electromagnetic energy
- (5.59b) $\chi(\bar{I}_{\rightarrow \bar{u}}^2) > 0$ { irreversible creation of thermal energy at the expense of electromagnetic energy (electric dissipation, or Joule's heat, due to conduction of current)
- (5.59c) $G_{pm} > 0$ { irreversible creation of thermal energy at the expense of electromagnetic energy (hysteresis heat, due to electric and magnetic polarization)
- (5.59d) $-(\bar{\Pi}_e \bar{g} \bar{r} \bar{a} \bar{d} \bar{u}) \geq 0$ { in the perfectly elastic body: reversible conversion of bond energy (or elastic energy) with kinetic energy
- (5.59e) $-\rho p \frac{dv}{dt} \bar{u} \geq 0$ { in the fluid: reversible conversion of thermal energy with kinetic energy (due to compression and expansion)
- (5.59f) $-(\bar{\Pi}_v \bar{g} \bar{r} \bar{a} \bar{d} \bar{u}) > 0$ { in the fluid: irreversible creation of thermal energy at the expense of kinetic energy (viscous dissipation)

There are three microphysical mechanisms of transport of internal energy: carrying with the grains of substance, transferring by means of the intra-molecular collisions, and transmitting in the form of the radiant energy, the emission and absorption of this energy being connected with the grain excitation.

In the phenomenologic approach, the first kind of transport is regarded as the convection and diffusion of internal energy. The diffusive transfer, occurring only in the multicomponent fluid mixture, will be discussed later (Sec. 6.11).

The second kind of transport is called the thermal conduction transfer, and the third one is the thermal radiation transfer (characterized by the wavelengths much smaller than of the shortest radio waves). Contrary to the first and second process, which occur in the substance-filled space only, the third one is of non-substantial character and may take place even in the vacuum.

Consequently, the non-convective transfer of internal energy consists of the diffusive, conductive, and radiative parts, their flux

densities being denoted by \bar{J}_d , \bar{J}_{tc} and \bar{J}_{tr} , respectively:

$$(5.60) \quad \bar{J}_{nc} = \bar{J}_d + \bar{J}_{tc} + \bar{J}_{tr}$$

In a homogenous and isotropic medium, and at the temperature field T of rather small variability in time and space, the vector \bar{J}_{tc} obeys the following equation:

$$(5.61) \quad \bar{J}_{tc} = -\lambda \overline{\text{grad}} T \quad \text{Fourier's law for the heat conduction}$$

where λ is called the thermal conductivity.

Exercise 5.18. Derive the following differential equation

$$(5.62) \quad \frac{\partial T}{\partial t} = a \operatorname{div} \overline{\text{grad}} T, \quad a = \frac{\lambda}{\rho c_v},$$

describing the process of heat conduction in the isotropic and homogenous substance ($\rho = \text{const}$, $\lambda = \text{const}$, the isochoric specific heat $c_v = \text{const}$), being at rest ($\bar{u} = 0$), and not subjected to electromagnetic phenomena (G_{pm} , $\bar{I} = 0$), radiation ($\bar{J}_{tr} = 0$), and diffusion ($\bar{J}_d = 0$).

Instructions. In the described medium, the variable part of internal energy is the function of temperature only ($\mathcal{E}_I = c_v T$). We use this property, as well as all the simplifying assumptions and Fourier's law (5.61) with the aim of transforming the balance (5.57). The obtained result (5.62) is the diffusion-type differential equation, and the coefficient a is called the thermal diffusivity (compare Eq.6.21 in the next chapter).

5.16. The First Law of Thermodynamics

The First Law of Thermodynamics states that the increment of internal energy within a closed system (i.e. substantial region) equals to the sum of the heat delivered to the system and the work performed on the system.

In the thermodynamics, there are several different definitions of the quantity called simply "work" with no adjective. None of tho-

se definitions coincides with the mechanical concept of "total work" expressed in Eq.(5.40b). To avoid confusion, let characterize three of the most frequently used thermodynamic definitions of work (the recognition adjectives being given by the present author).

The "statically-potential work" is a sum of those energetic processes that are due to the action of mechanical forces and can cause lasting and reversible changes of the potential energy of the balance system or another system interacting with the first one, at unchanged kinetic energy of both systems. The last condition excludes the "mechanical work" equal to the increment of kinetic energy (Eqs.5.40b, 5.41). The work of frictional forces (viscosity) is not taken into account because of the irreversible character of dissipation. Consequently, the occurrence of the "statically-potential work" is restricted to the phenomena of strain, compression, or expansion (compare Ex. 5.15):

$$(5.63a) \quad \left. \begin{array}{l} \text{"statically-potential work"} \\ \text{per unit mass of the system,} \\ \text{performed during time } dt \end{array} \right\} = \begin{cases} -v(\bar{\rho}_e \bar{g} \bar{r} \bar{a} \bar{d} \bar{u} \, dt) & \text{for elastic body} \\ -p \, \underline{dv}_{\bar{u}} & \text{for fluid} \end{cases}$$

In the thermodynamics, the attention is sometimes paid to the dualistic character of frictional dissipation which has some common features with both work and heat. Truesdell (see Bibliography 7) has included the viscous dissipation in the so-called "net working" which is equal to the volume integral of the second term on the right-hand side of Eq.(5.40b). This way the "net work" (being the time integral of "net working") fits to the definition of "thermodynamic work", given in Sec. 5.9:

$$(5.63b) \quad \left. \begin{array}{l} \text{"net work" per unit mass} \\ \text{of the system, performed} \\ \text{during time } dt \end{array} \right\} = -v(\bar{\rho} \bar{g} \bar{r} \bar{a} \bar{d} \bar{u} \, dt) \quad \begin{array}{l} \text{for any kind} \\ \text{of substance} \end{array}$$

The third thermodynamic definition of work extends this notion over the non-mechanical phenomena. The "generalized work" is supposed to be a sum of integrals of various "generalized forces" along

"generalized displacements". This concept contains not only the work of various mechanical forces but also the "work of electric current", the "work of charging a condenser", the "work of magnetic hysteresis", etc. Whether a particular process belongs to work or heat (notice the dualism of the "works" of electric current and hysteresis!), it is usually decided by the author of textbook in an arbitrary manner.

In the thermodynamics, the heat is supposed to be all the energetic processes causing the change of internal energy and not being the work. Since there are different definitions of work, they must correspond to different definitions of heat.

While each of the three kinds of "work" used in thermodynamics is in fact a production term of the internal energy balance, the heat is a sum of terms of production (the "internal heat" developed within the system) and transfer (the "external heat" delivered through the boundaries of the system). Thus the concepts of work and heat do not coincide with the balance terminology. In order to reveal this discrepancy, we shall interpret the balance of internal energy (5.57a) (multiplied by $v dt$, and shown in two variants of the elastic body and the fluid) in terms of the First Law of Thermodynamics, at three different definitions of work and heat:

$$\begin{aligned}
 (5.64) \quad \frac{d\xi_I}{dt} = & \left\{ \underbrace{-v(\bar{\rho}_e \bar{g} \bar{r} \bar{a} \bar{d} \bar{u} dt)}_{\text{"statically-potential work"}} - \underbrace{p dv}_{\text{"net work"}} - v(\bar{\rho}_v \bar{g} \bar{r} \bar{a} \bar{d} \bar{u} dt) \right\} + \underbrace{v G_{pm} dt}_{\text{heat}} + \underbrace{v(\bar{E} \bar{I}) dt}_{\text{heat}} - \underbrace{v \operatorname{div} \bar{J}_{nc} dt}_{\text{heat}}
 \end{aligned}$$

In the classical thermodynamics, the First Law is supposed to be an energetic balance ranking among universal laws of Nature. Its role, however, seems to be performed better by the balance of internal energy (5.57a). Owing to the composite approach to various

physical phenomena, Eq.(5.57a) needs not to be postulated. It has been derived from the principle of conservation of total energy and precisely specified balances of the kinetic, gravitational and electromagnetic energies. In such a presentation, the vague concepts of work and heat have been replaced with rigorous forms of conversion and transfer of energy.

5.17. The Balance of Enthalpy of the Fluid

The fluid substance is characterized by a thermodynamic function of state called the enthalpy (or static enthalpy, strictly). This is a scalar SQ of the same dimension as the energy. Its storage mass-density H and volume-density ϱH are defined as follows:

$$(5.65a) \quad H = \varepsilon_I + pv,$$

$$(5.65b) \quad \varrho H = \varrho \varepsilon_I + p$$

Noticing that

$$(5.66) \quad \varrho \frac{d(pv)}{dt} \vec{u} = \varrho p \frac{dv}{dt} \vec{u} + \frac{dp}{dt} \vec{u},$$

adding Eqs.(5.58a) and (5.66), and taking into consideration the definition (5.65a), we obtain the balance of enthalpy at a substantial point:

$$(5.67) \quad \varrho \frac{dH}{dt} \vec{u} = G_{pm} + (\vec{E} \cdot \vec{I}_{\vec{u}}) + \frac{dp}{dt} \vec{u} - (\vec{\Pi}_v \vec{g} \vec{r} \vec{a} \vec{d} \vec{u}) - \text{div } \vec{J}_{nc}$$

From comparison of balances (5.67) and (5.58a) we conclude that the non-convective transfer of enthalpy is identical with that of the internal energy, the production being different in both cases. Though the enthalpy has the dimension of energy, it is not any specific kind of energy. If we replaced the internal energy with the enthalpy and added the kinetic, gravitational and electromagnetic energies, the obtained sum would not satisfy the conservation principle.

Exercise 5.19. Prove that in an isentropic and isobaric process the specific enthalpy H remains unchanged along a substantial displacement.

Instructions. In an isentropic process, the phenomena of hysteresis, conduction of current, viscosity, diffusion, heat conduction and radiation do not occur ($G_{pm}, \bar{I}_{\vec{u}}, \bar{I}_{\vec{v}}, \bar{J}_{nc} = 0$). Considering all this in the balance (5.67), we obtain the relation

$$(5.68) \quad \rho \frac{dH_{\vec{u}}}{dt} = \frac{dp_{\vec{u}}}{dt}, \text{ or } dH_{\vec{u}} = v dp_{\vec{u}} \quad \text{for isentropic process}$$

In an isobaric process, $dp_{\vec{u}} = 0$ along the substantial displacement, so $dH_{\vec{u}} = 0$.

6. The Balances of Componential Quantities

6.1. Composition of Mixture. Production of the Component Mass

The multicomponent mixture is a set of the substance components which coexist within the same geometric region. In case of the multicomponent fluid, the components can move differently and are mutually convertible.

The mixture may be regarded as a continuum also in case when it contains a sufficiently granulated phase, provided it satisfies the condition of phenomenological averaging (2.2b). The aerosol (dust-filled gas) and the wet steam (gas filled with liquid droplets) are examples of such phase-mixture. According to other criteria, we distinguish the mixtures of chemical components (e.g. a liquid solution) and of electric-charge components (plasma, electrolyte).

Several properties of the mixture as a whole (denoted without component's suffix) may be expressed as a sum of the respective properties of all n components. Since we shall deal with the sums within closed sets of components only, the sum symbol \sum will be deprived of the usual complementary denotation: "from $i = 1$ to n ".

The composition of mixture is defined by the dimensionless fractions, or concentrations, of particular components with respect to the total amount of mixture's substance, which may be computed on three different bases: the stored mass, the occupied volume, or the

number of moles. In this present abridged exposition, we shall use neither the volume- nor the mole-fractions, basing our considerations solely on the fundamental concept of the mass-fraction, or mass-concentration:

$$(6.1) \quad \left. \begin{array}{l} \text{mass-fraction} \\ \text{of } i\text{-th component} \end{array} \right\} = \limf \frac{\text{stored } i\text{-th mass}}{\text{stored mixture mass}} = c_i$$

Owing to phase-reaction (e.g. vaporization of liquid droplets in gas), chemical reaction, or electric reaction (ionization, recombination), the mass of i -th component can be created or annihilated, its production being characterized by:

$$(6.2) \quad \left. \begin{array}{l} \text{mass-density of} \\ i\text{-th mass production} \end{array} \right\} = \limf \frac{\text{produced } i\text{-th mass}}{\text{time} \cdot \text{stored mixture mass}} = \xi_i$$

Comparing definitions (6.1), (6.2), (5.7), (5.8), we conclude that the i -th mass may be regarded as a scalar SQ which obeys the relations (5.9a), (5.10a), with $\varepsilon = c_i$, $\mu = \xi_i$, $\varphi = \varrho_i$, $\Psi = \varrho \xi_i$ substituted. Consequently, the mass-fraction is the quotient of the storage densities of masses of the i -th component (2.5b) and of the mixture as a whole (2.5a):

$$(6.3) \quad c_i = \varrho_i / \varrho,$$

and the volume-density of production of i -th mass is $\varrho \xi_i$.

Since within any region the sum of masses of all the components forming closed set must be equal to mass of the mixture, then, complying with Eq.(6.3), we have

$$(6.4a) \quad \sum \varrho_i = \varrho, \quad (6.4b) \quad \sum c_i = 1$$

It results from the principle of conservation of the total mass that the sum of production densities of all the component masses must be zero:

$$(6.5a) \quad \sum \varrho \xi_i = 0, \quad \sum \xi_i = 0$$

Complying with this postulate, the creation ($\xi_i > 0$) of some components must be accompanied with the annihilation ($\xi_i < 0$) of other components.

In a simple reaction process we distinguish three groups of components: the products being created ($\xi_P > 0$), the reactants being annihilated ($\xi_S < 0$), and the neutral components not participating in reaction ($\xi_N = 0$). According to postulate (6.5b), the rate of reaction is defined as

$$(6.6) \quad \xi_P = -\xi_S > 0$$

It has to be emphasized that the above increment of concentration of products per unit time is due solely to creation, and not to the transfer. In many textbooks on chemical kinetics we find the rate of reaction expressed in form of the derivative dc_P/dt ; this is true only for the concentration c_P averaged over the whole vessel, provided its walls being impermeable for any component of substance.

6.2. Motion of the Substance Component. Diffusion of Mass

According to definition (2.6b), the i -th mass is moving with the same componential velocity \vec{u}_i as that of the componential surface \vec{s}_i (Sec. 3.4). Consequently, the transfer-flux density of i -th mass with respect to the i -th surface must be zero:

$$(6.7a) \quad \vec{j}_{i \rightarrow \vec{u}_i} = 0$$

Using this in Eq.(4.20a) at $\vec{w} = \vec{u}_i$, we find that the i -th mass-flux density through immobile surface is

$$(6.7b) \quad \vec{j}_i = \rho_i \vec{u}_i$$

Applying this once again to the rule (4.20a), we obtain the i -th mass-flux density through referential surface:

$$(6.7c) \quad \vec{j}_{i \rightarrow \vec{w}} = \rho_i (\vec{u}_i - \vec{w})$$

Since the sum of masses of all the components flown through immobile surface must be equal to the transferred mass of mixture as a whole, then, with the help of Eqs.(6.7b) and (5.1b), we have

$$(6.8a) \quad \sum \bar{J}_i = \bar{J}, \quad (6.8b) \quad \sum \varrho_i \bar{u}_i = \varrho \bar{u}$$

Substitution of Eq.(6.3) into Eq.(6.8b) gives the barycentric velocity as the mass-weighted average of all the componential velocities:

$$(6.8c) \quad \bar{u} = \sum c_i \bar{u}_i$$

There is also the concept of volume-centric velocity of mixture, being the weighted average with respect to volume-fractions of components.

As far as the balance is concerned, particular attention must be paid to the motion of substance's component with respect to substantial point, surface, and region, called the diffusive motion. The i -th mass-flux density through substantial surface \bar{J}_i is called the diffusion-flux density. Since this quantity is of a $\xrightarrow{\bar{u}}$ very frequent use, we shall replace its inconvenient arrow-symbol with traditional denotation \bar{J}_{di} , bearing in mind that it describes the transfer referred to the substantial surface. Substituting $\bar{w} = \bar{u}$ into Eq.(6.7c) and using Eqs.(6.7b), (6.3), (5.1b), we obtain the expression for \bar{J}_i

$$(6.9) \quad \bar{J}_{di} = \bar{J}_{i \xrightarrow{\bar{u}}} = \varrho_i (\bar{u}_i - \bar{u}) = \bar{J}_i - \varrho_i \bar{u} = \bar{J}_i - c_i \bar{J}$$

It results from Eqs.(6.8a) and (6.4b) that

$$(6.10) \quad \sum \bar{J}_{di} = 0$$

Rearranging Eq.(6.9), we express \bar{J}_i as the sum of two terms:

$$(6.11) \quad \bar{J}_i = \underbrace{\varrho_i \bar{u}}_{\text{contribution to transfer of total mass}} + \underbrace{\bar{J}_{di}}_{\text{mutual mixing of components' masses}}$$

Fig. 6.1 For each component, the first term is a vector directed identically as the barycentric velocity, and so contributing to transfer of the total mass. The second term describes solely the pure mixing of components, because the sum of all \bar{J}_{di} gives zero. Summing up Eq.(6.11) over all the components, we obtain: $\bar{J} = \varrho \bar{u} + 0$. This result is explained by the example of binary diffusion (in the 2-component-mixture, $i = 1, 2$) illustrated in Fig. 6.1, where \bar{J}_{d1} and $\bar{J}_{d2} = -\bar{J}_{d1}$ are opposite vectors.

If the mixture as a whole is at rest, Eq.(6.9) takes particular form:

$$(6.12) \quad \bar{J}_i = \varrho_i \bar{u}_i = \bar{J}_{di} \quad \text{for resting mixture } (\bar{u} = 0)$$

Various factors may cause the phenomenon of mass-diffusion. At small variability of the fields c_i , the vector \bar{J}_{di} obeys the following law:

$$(6.13) \quad \bar{J}_{di} = -\varrho D \overline{\text{grad}} c_i \quad \left\{ \begin{array}{l} \text{Fick's law of the free diffusion} \\ \text{in the binary } (i = 1, 2) \text{ incom-} \\ \text{pressible } (\varrho = \text{const}) \text{ mixture} \end{array} \right.$$

where $D = \text{const}$ is called the diffusivity.

Exercise 6.1. Distinguish the convective and non-convective transfer of i -th mass in Eq.(6.11).

Instructions. Apply substitution (6.3) and compare with the classifying decomposition (5.13a). Since the i -th mass is a scalar SQ, both expressions will prove equivalent at substitutions: $\bar{H} = \bar{J}_i$, $\bar{H}_{\rightarrow \bar{u}} = \bar{J}_{i \rightarrow \bar{u}} = \bar{J}_{di}$, $\varepsilon = c_i$.

Exercise 6.2. Check whether the right-hand side of Fick's law (6.13) satisfies Eq.(6.10).

Instruction. Use Eq.(6.4b) with $i = 1, 2$ substituted.

6.3. The Balance of the Component Mass

If we regard the i -th mass as a scalar EQ, its referential balances result from general Eqs.(4.28a) and (4.30a), in which we

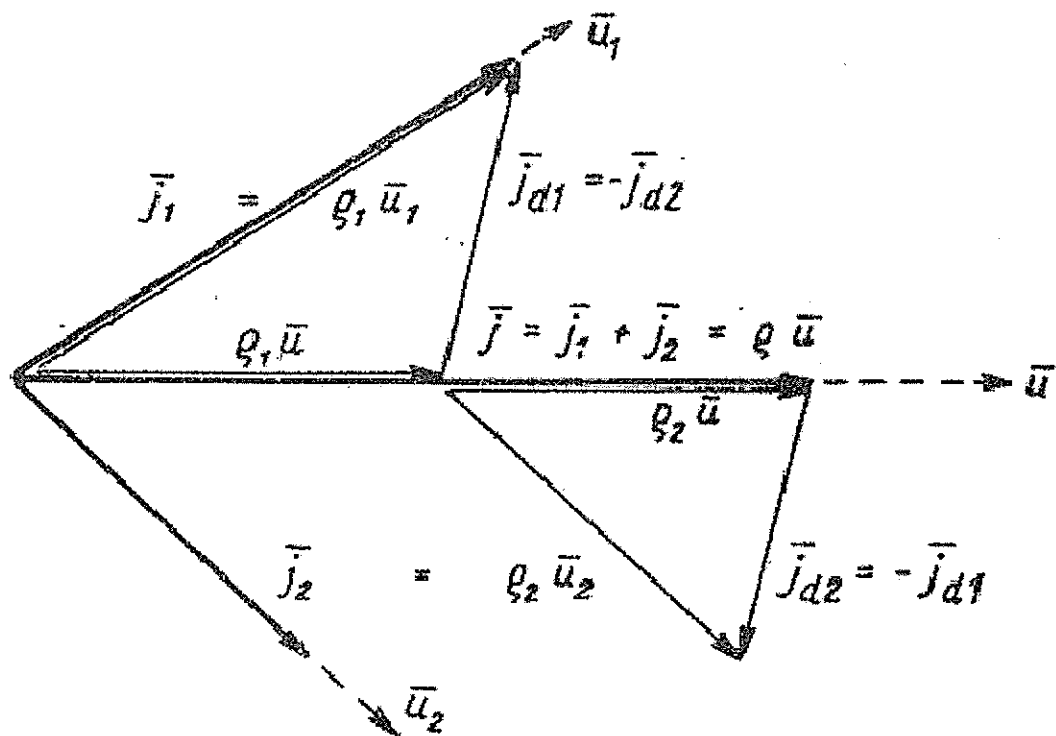


Fig. 6.1. Graph of mass transfer vectors in binary diffusion

replace ψ with ρ_i , Ψ with ρ_i^E , and $\bar{H}_{\bar{w}}$ with expression (6.7c):

$$(6.14a) \quad \frac{d}{dt} \int_{\bar{V}_{\bar{w}}} \rho_i d^3V = \int_{\bar{V}_{\bar{w}}} \rho_i^E d^3V - \oint_{\bar{S}_{\bar{w}}} (\rho_i \{\bar{u}_i - \bar{w}\} d^2s)$$

$$(6.14b) \quad \text{reyn}_{\bar{w}} \rho_i = \frac{d\rho_i}{dt} \rightarrow \bar{w} + \rho_i \text{div } \bar{w} = \rho_i^E - \text{div}(\rho_i \{\bar{u}_i - \bar{w}\})$$

At $\bar{w} = 0$, we obtain the balances for immobile region and point:

$$(6.15a) \quad \frac{\partial}{\partial t} \int_V \rho_i d^3V = \int_V \rho_i^E d^3V - \oint_S (\rho_i \bar{u}_i d^2s)$$

$$(6.15b) \quad \frac{\partial \rho_i}{\partial t} = \rho_i^E - \text{div}(\rho_i \bar{u}_i), \text{ or } (6.15c) \quad \frac{\partial \rho_i}{\partial t} + \text{div}(\rho_i \bar{u}_i) = \rho_i^E$$

Substituting $\bar{w} = \bar{u}$ in Eqs.(6.14) and taking into consideration Eqs.(6.3), (6.9), (5.11a), we obtain the substantial balances, Eq. (6.16b) being sometimes called the continuity equation of diffusion:

$$(6.16a) \quad \frac{d}{dt} \int_{\bar{V}_{\bar{u}}} \rho c_i d^3V = \int_{\bar{V}_{\bar{u}}} \rho_i^E d^3V - \oint_{\bar{S}_{\bar{u}}} (\bar{J}_{di} d^2s)$$

$$(6.16b) \quad \text{reyn}_{\bar{u}}(\rho c_i) = \rho \frac{dc_i}{dt} \rightarrow \bar{u} = \rho_i^E - \text{div } \bar{J}_{di}$$

Finally, if we substitute $\bar{w} = \bar{u}_i$ into Eqs.(6.14), we obtain the componential balances, characterized by the absence of any transfer (the change of i -th mass stored within i -th componential region can result from production, i.e. reaction, only):

$$(6.17a) \quad \frac{d}{dt} \int_{\bar{V}_{\bar{u}_i}} \rho_i d^3V = \int_{\bar{V}_{\bar{u}_i}} \rho_i^E d^3V, \quad (6.17b) \quad \frac{d\rho_i}{dt} \rightarrow \bar{u}_i + \rho_i \text{div } \bar{u}_i = \rho_i^E$$

Comparing Eqs.(6.15c) with (5.4d) and (6.17b) with (5.4b), we find general similarity of structure, with the exception of zero-valued right-hand sides for the total mass, which is a conservative quantity.

Exercise 6.3. Derive the balances (6.16a,b) from general Eqs. (5.18a) and (5.19a).

Instructions. Regard the i -th mass as a scalar SQ, motivating the substitutions $\xi = c_i$, $\mu = \xi_i$, $\vec{H}_{\vec{u}} = \vec{J}_{di}$ (see Ex. 6.1).

Exercise 6.4. Prove that in case of each of Eqs.(6.14a,b), (6.15a,c), (6.16a), the sum of balances over all i -th components becomes the balance of total mass, expressed by Eqs.(5.2a,b), (5.3b) (5.4d), (5.3a), respectively.

Instruction. Use Eqs.(6.4a,b), (6.5a), (6.8b) and (6.10).

Exercise 6.5. Prove that the sum of Eq.(6.16b) over all the components is a zero identity.

Instruction as for Ex. 6.4.

6.4. Some Exemplary Interpretations and Applications of the Component Mass Balance

Using the balance (6.17b) and Eq.(6.3), we derive a dimensionless equation expressing the relative increment of density ϱ_i along the componential displacement $\vec{u}_i dt$ of the i -th mass element during time dt :

$$(6.18) \quad \underbrace{\frac{d\varrho_i}{\varrho_i} \rightarrow \vec{u}_i}_{\substack{\text{relative increment} \\ \text{of density along} \\ \text{componential displacement}}} = \underbrace{\frac{\xi_i dt}{c_i}}_{\substack{\text{relative increment} \\ \text{of concentration} \\ \text{due to production} \\ \text{(reaction)}}} - \underbrace{\text{div } \vec{u}_i dt}_{\substack{\text{relative increment} \\ \text{of volume of} \\ \text{componential region}}}$$

Fig. 6.2 The first cause of change of ϱ_i is creation or annihilation of the component due to reaction (e.g. conversion of the "white" grains into the "black" ones in Fig. 6.2). The second cause lies in the dilatation of the componential region, which is equal to expression (3.34) at $\vec{w} = \vec{u}_i$, multiplied by dt (the minus sign explains that the contraction of region, $\text{div } \vec{u}_i < 0$, contributes to the increase of ϱ_i , as it is shown in Fig. 6.2). In terms of Eq.(6.18), the change of ϱ_i is independent of diffusion which does not occur through the shell of componential region.

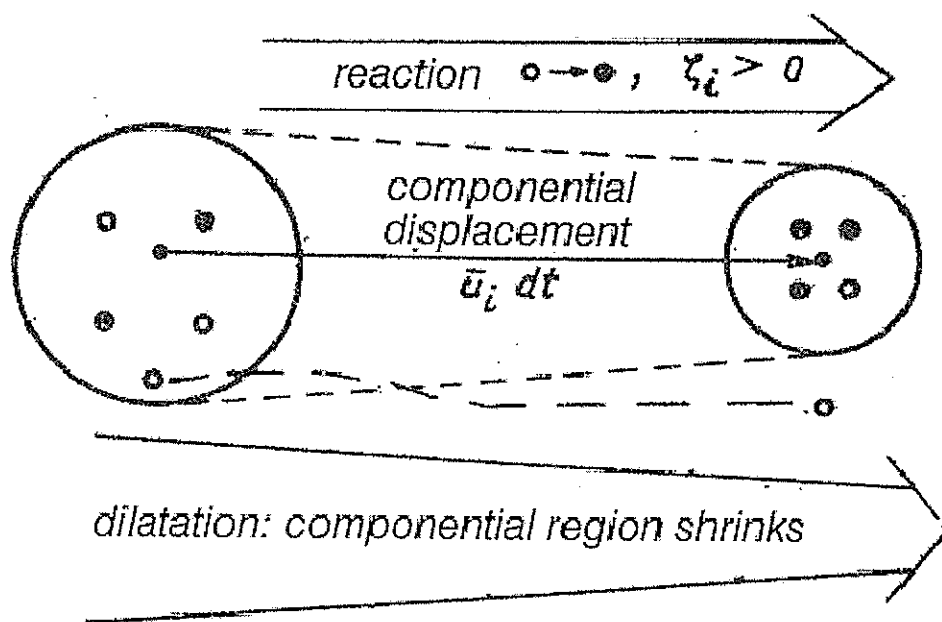


Fig. 6.2. Influence of reaction and dilatation on the density of the "i-th" mass ("black" grains)

Multiplying Eq.(6.16b) by $v dt = dt/\varrho$, we obtain the increment of concentration c_i along the substantial displacement $\vec{u} dt$:

$$(6.19) \quad \underbrace{\frac{dc_i}{\rightarrow \vec{u}}}_{\substack{\text{increment of} \\ \text{concentration} \\ \text{along substantial} \\ \text{displacement}}} = \underbrace{\sum_i dt}_{\substack{\text{increment due to} \\ \text{production (reaction)}}} - \underbrace{v \operatorname{div} \vec{j}_{di} dt}_{\substack{\text{increment due to} \\ \text{transfer (diffusion)}}$$

The causes of increment of c_i are: production of i -th mass (reaction) and diffusion, as the only possible kind of i -th mass transfer through substantial shell s_u . In terms of Eq.(6.19), the dilatation of V_u does not influence the concentration c_i as the quotient of masses stored within the same volume.

Fig. 6.3 The significance of both terms of Eq.(6.19) will be explained by way of example of the combustion reaction (Fig. 6.3). In the perfectly kinetic combustion, the combustible mixture (the reactants) is properly composed before the onset of reaction, and the diffusion is of no influence. This way, in the course of combustion progress, the only process occurring within the substantial region is the conversion of reactants into products (until the complete exhaustion of the reactants):

$$(6.20a) \quad \frac{dc_i}{\rightarrow \vec{u}} = \sum_i dt, \quad \operatorname{div} \vec{j}_{di} = 0 \quad \text{for perfectly kinetic combustion}$$

In the diffusive combustion, the proper forming of the combustible mixture is achieved by means of diffusion, which occurs simultaneously with the reaction (the oxidant is entering, while the burnt gas is leaving the substantial region). In case of the perfectly diffusive combustion, it is assumed that the concentrations c_i remain constant along the substantial displacement, because the production is entirely compensated by the diffusive transfer:

$$(6.20b) \quad \sum_i dt = v \operatorname{div} \vec{j}_{di} dt, \quad \frac{dc_i}{\rightarrow \vec{u}} = 0 \quad \left\{ \begin{array}{l} \text{for perfectly} \\ \text{diffusive combustion} \end{array} \right.$$

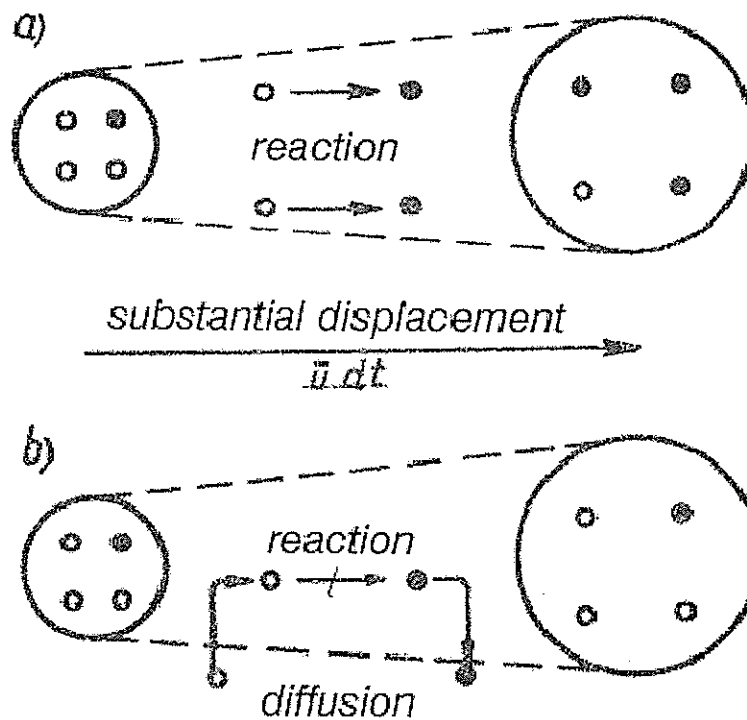


Fig. 6.3. Influence of reaction and diffusion on the concentration of products ("black" grains) in two cases of combustion

(a) perfect kinetic combustion $\text{div } \vec{j}_{\text{di}} = 0$

(b) perfect diffusive combustion $dc_i = 0$
 $\rightarrow \vec{u}$

Exercise 6.6. Derive the diffusion-type differential equation

$$(6.21) \quad \frac{\partial c_i}{\partial t} = D \operatorname{div} \overline{\operatorname{grad} c_i} \quad \left\{ \begin{array}{l} \text{for free diffusion in the binary} \\ \text{non-reacting incompressible resting} \\ \text{mixture} \end{array} \right.$$

Instructions. Set $\xi_i = 0$ into the balance (6.15b), apply substitutions (6.3), (6.12), (6.13) and assumption (5.5b). Discuss the analogy between phenomena of diffusion and heat conduction (compare Eqs. 6.21 and 5.62) and between coefficients D and \underline{a} .

6.5. The Componential Quantity

If the mixture is the site of a certain SQ, then, at the fulfillment of some physical postulates, we can decompose this quantity into "sub-substantial" parts allotted to particular components.

The componential quantity, or CQ, as abbreviated, is an extensive quantity which can be stored in a component of substance and carried with this component. The CQ needs not to be a conservative quantity, and the carrying with the component is not, in general, the only way of CQ's transport.

Since the i -th mass is a measure of the site of i -th CQ, the storage of CQ depends on the component mass stored within the balance region. The continuous CQs are characterized by the following partial densities:

Category of the partial density	Definition	Denotation for	
		scalar CQ	vector CQ
(6.22) storage partial density of i -th CQ	$= \limf \frac{\text{stored } i\text{-th CQ}}{\text{stored } i\text{-th mass}}$	ϵ_i	\bar{Q}_i
(6.23) production partial density of i -th CQ		μ_i	\bar{S}_i

Comparing definitions (2.5b), (4.5), (4.6), (6.22) and (6.23), and considering Eq.(6.3), we obtain the relations between the volume-densities of storage φ_i , \bar{P}_i and production ψ_i , \bar{R}_i of i -th CQ,

and the respective partial densities:

$$(6.24a) \quad \varphi_i = \varrho_i \varepsilon_i = \varrho c_i \varepsilon_i, \quad (6.24b) \quad \bar{\varphi}_i = \varrho_i \bar{Q}_i = \varrho c_i \bar{Q}_i$$

$$(6.25a) \quad \psi_i = \varrho_i \mu_i = \varrho c_i \mu_i, \quad (6.25b) \quad \bar{\psi}_i = \varrho_i \bar{S}_i = \varrho c_i \bar{S}_i$$

It is possible to derive (see Ex. 6.7 to 6.9) the following transformations of the Reynoldsians and relations between the componential and substantial derivatives:

$$(6.26a) \quad \begin{aligned} \text{reyn}(\varrho_i \varepsilon_i) &= \frac{\partial(\varrho_i \varepsilon_i)}{\partial t} + \text{div}(\bar{u}_i \varepsilon_i \varrho_i) = \\ &\quad \text{I form} \\ &= \varrho \frac{d(c_i \varepsilon_i)}{dt} + \text{div}(\bar{j}_{di} \varepsilon_i) = \frac{d(\varrho_i \varepsilon_i)}{dt} + \varrho_i \varepsilon_i \text{div} \bar{u}_i = \varrho_i \frac{d\varepsilon_i}{dt} + \varrho_i \varepsilon_i \text{div} \bar{u}_i \\ &\quad \text{II form} \qquad \qquad \qquad \text{III form} \qquad \qquad \qquad \text{IV form} \end{aligned}$$

$$(6.26b) \quad \begin{aligned} \text{reyn}(\varrho_i \bar{Q}_i) &= \frac{\partial(\varrho_i \bar{Q}_i)}{\partial t} + \text{div}\{\bar{u}_i \bar{Q}_i \varrho_i\} = \\ &\quad \text{I form} \\ &= \varrho \frac{d(c_i \bar{Q}_i)}{dt} + \text{div} \bar{j}_{di} \bar{Q}_i = \frac{d(\varrho_i \bar{Q}_i)}{dt} + \varrho_i \bar{Q}_i \text{div} \bar{u}_i = \varrho_i \frac{d\bar{Q}_i}{dt} + \varrho_i \bar{Q}_i \text{div} \bar{u}_i \\ &\quad \text{II form} \qquad \qquad \qquad \text{III form} \qquad \qquad \qquad \text{IV form} \end{aligned}$$

$$(6.27a) \quad \text{reyn}(\varrho_i \varepsilon_i) = \text{reyn}(\varrho c_i \varepsilon_i) = \varrho \frac{d(c_i \varepsilon_i)}{dt}$$

$$(6.27b) \quad \text{reyn}(\varrho_i \bar{Q}_i) = \text{reyn}(\varrho c_i \bar{Q}_i) = \varrho \frac{d(c_i \bar{Q}_i)}{dt}$$

$$(6.28a) \quad \varrho_i \left\{ \frac{d\alpha}{dt} - \frac{d\alpha}{dt} \right\} = (\bar{j}_{di} \text{grad } \alpha)$$

$$(6.28b) \quad \varrho_i \left\{ \frac{d\bar{A}}{dt} - \frac{d\bar{A}}{dt} \right\} = \bar{j}_{di} \text{grad } \bar{A}$$

Exercise 6.7. Prove the transformations (6.26a,b).

Instructions. Set $\bar{w} = \bar{u}_i$ into expressions (3.25a,b), then substitute $\alpha = \varphi_i$ (6.24a) and $\bar{A} = \bar{\varphi}_i$ (6.24b), in order to obtain I and III form. Apply substitutions (6.9) and (6.3) and transformations (5.11a,b) at $\varepsilon = c_i \varepsilon_i$, $\bar{Q} = c_i \bar{Q}_i$ into I form in order to obtain II form. Develop the derivative of product and use the balance (6.17b) in III form, in order to obtain IV form.

Exercise 6.8. Prove the transformations (6.27a,b).

Instruction. Substitute $\varepsilon = c_i \varepsilon_i$, $\bar{Q} = c_i \bar{Q}_i$ into transformations (5.11a,b) and use Eq.(6.3).

Exercise 6.9. Prove the relations (6.28a,b).

Instruction. Use Eqs.(3.4a,b) and (6.9).

6.6. Diffusive and Non-Diffusive Transfer of a Componential Quantity

Substituting $\varphi = \varphi_i$ (6.24a), $\bar{P} = \bar{P}_i$ (6.24b) into Eqs.(4.20), firstly at $\bar{w} = \bar{u}$ and secondly at $\bar{w} = \bar{u}_i$, and considering Eq.(6.9), we have

$$\begin{array}{l} (6.29a) \quad \bar{H}_i = \bar{u} \varepsilon_i \varphi_i + \bar{H}_{i \rightarrow \bar{u}} = \bar{u}_i \varepsilon_i \varphi_i + \bar{H}_{i \rightarrow \bar{u}_i} \\ (6.29b) \quad \bar{T}_i = \bar{u} \bar{Q}_i \varphi_i + \bar{T}_{i \rightarrow \bar{u}} = \bar{u}_i \bar{Q}_i \varphi_i + \bar{T}_{i \rightarrow \bar{u}_i} \end{array} \quad \left\{ \begin{array}{l} (6.30a) \quad \bar{H}_{i \rightarrow \bar{u}} - \bar{H}_{i \rightarrow \bar{u}_i} = \bar{j}_{di} \varepsilon_i \\ (6.30b) \quad \bar{T}_{i \rightarrow \bar{u}} - \bar{T}_{i \rightarrow \bar{u}_i} = \bar{j}_{di} \bar{Q}_i \end{array} \right.$$

and finally we obtain the classifying decomposition of the transfer-flux density of CQ with respect to immobile surface:

$$\begin{array}{l} (6.31a) \quad \bar{H}_i = \underbrace{\bar{u} \varepsilon_i \varphi_i}_{\text{convective}} + \underbrace{\bar{j}_{di} \varepsilon_i + \bar{H}_{i \rightarrow \bar{u}_i}}_{\text{non-convective}} \\ (6.31b) \quad \bar{T}_i = \underbrace{\bar{u} \bar{Q}_i \varphi_i}_{\text{convective}} + \underbrace{\bar{j}_{di} \bar{Q}_i + \bar{T}_{i \rightarrow \bar{u}_i}}_{\text{non-convective}} \end{array}$$

transfer: convective diffusive non-diffusive / non-convective

In the convective and diffusive transfer, the CQ is carried with the mixture as a whole (moving with barycentric velocity) and with the component (in its diffusive motion), respectively. The diffusive transfer is the capture of CQ's storage resulting from the relative motion of componential surface $s_{\rightarrow \bar{u}_i}$ with respect to the substantial surface $s_{\rightarrow \bar{u}}$.

The non-diffusive transfer is directly referred to the componential surface, so it is connected with the motion of neither mixture nor its components. In terms of classification (5.13), the sum of the diffusive and non-diffusive transfers forms the non-convective transfer.

Substituting $\varphi = \varphi_i$ (6.24a), $\bar{P} = \bar{P}_i$ (6.24b), $\bar{H} = \bar{H}_i$ (6.31a), $\bar{T} = \bar{T}_i$ (6.31b) into Eqs.(4.20), we obtain the transfer-flux densities of CQ through referential surface:

$$(6.32a) \quad \bar{H}_{i \rightarrow \bar{w}} = \bar{H}_i - \bar{w} \varepsilon_i \rho_i = \bar{H}_{i \rightarrow \bar{u}_i} + \bar{j}_{di} \varepsilon_i - (\bar{w} - \bar{u}) \varepsilon_i \rho_i$$

$$(6.32b) \quad \bar{T}_{i \rightarrow \bar{w}} = \bar{T}_i - \bar{w} \bar{Q}_i \rho_i = \bar{T}_{i \rightarrow \bar{u}_i} + \bar{j}_{di} \bar{Q}_i - (\bar{w} - \bar{u}) \bar{Q}_i \rho_i$$

6.7. The Basic Balances of a Componential Quantity

Substituting $\varphi = \varphi_i$ (6.24a), $\bar{P} = \bar{P}_i$ (6.24b), $\psi = \psi_i$ (6.25a), $\bar{R} = \bar{R}_i$ (6.25b), $\bar{H}_{\rightarrow \bar{w}} = \bar{H}_{i \rightarrow \bar{w}}$ (6.32a), $\bar{T}_{\rightarrow \bar{w}} = \bar{T}_{i \rightarrow \bar{w}}$ (6.32b) into Eqs.(4.28) and (4.30), we obtain the referential balances of CQs:

$$(6.33a) \quad \frac{d}{dt} \int_{V_{\rightarrow \bar{w}}} \rho_i \varepsilon_i d^3V = \int_{V_{\rightarrow \bar{w}}} \rho_i \mu_i d^3V - \oint_{S_{\rightarrow \bar{w}}} d^2s \left\{ \bar{H}_{i \rightarrow \bar{u}_i} + \bar{j}_{di} \varepsilon_i - (\bar{w} - \bar{u}) \varepsilon_i \rho_i \right\}$$

$$(6.33b) \quad \frac{d}{dt} \int_{V_{\rightarrow \bar{w}}} \rho_i \bar{Q}_i d^3V = \int_{V_{\rightarrow \bar{w}}} \rho_i \bar{S}_i d^3V - \oint_{S_{\rightarrow \bar{w}}} d^2s \left\{ \bar{T}_{i \rightarrow \bar{u}_i} + \bar{j}_{di} \bar{Q}_i - (\bar{w} - \bar{u}) \bar{Q}_i \rho_i \right\}$$

$$(6.34a) \quad \text{reyn}_{\rightarrow \bar{w}}(\rho_i \varepsilon_i) = \frac{d(\rho_i \varepsilon_i)}{dt} \rightarrow \bar{w} + \rho_i \varepsilon_i \text{div } \bar{w} = \\ = \rho_i \mu_i - \text{div} \left\{ \bar{H}_{i \rightarrow \bar{u}_i} + \bar{j}_{di} \varepsilon_i - (\bar{w} - \bar{u}) \varepsilon_i \rho_i \right\}$$

$$(6.34b) \quad \text{reyn}_{\rightarrow \bar{w}}(\rho_i \bar{Q}_i) = \frac{d(\rho_i \bar{Q}_i)}{dt} \rightarrow \bar{w} + \rho_i \bar{Q}_i \text{div } \bar{w} = \\ = \rho_i \bar{S}_i - \text{div} \left\{ \bar{T}_{i \rightarrow \bar{u}_i} + \bar{j}_{di} \bar{Q}_i - (\bar{w} - \bar{u}) \bar{Q}_i \rho_i \right\}$$

Substituting $\bar{w} = \bar{u}_i$, \bar{u} , 0 in succession and considering Eqs. (6.9), (6.26), and (6.27), we obtain the balances for componential, substantial and immobile region/point, respectively:

$$(6.35a) \quad \frac{d}{dt} \int_{\vec{u}_i} \varrho_i \varepsilon_i d^3v = \int_{\vec{u}_i} \varrho_i \mu_i d^3v - \oint_{\vec{u}_i} (d^2s \cdot \vec{H}_i)_{\vec{u}_i}$$

$$(6.35b) \quad \frac{d}{dt} \int_{\vec{u}_i} \varrho_i \bar{Q}_i d^3v = \int_{\vec{u}_i} \varrho_i \bar{S}_i d^3v - \oint_{\vec{u}_i} \overline{d^2s \cdot \vec{T}_i}_{\vec{u}_i}$$

$$(6.36a) \quad \frac{d(\varrho_i \varepsilon_i)}{dt} \int_{\vec{u}_i} + \varrho_i \varepsilon_i \operatorname{div} \vec{u}_i = \varrho_i \frac{d\varepsilon_i}{dt} \int_{\vec{u}_i} + \varrho_i^e \varepsilon_i = \varrho_i \mu_i - \operatorname{div} \vec{H}_i \int_{\vec{u}_i} \\ = \operatorname{reyn}(\varrho_i \varepsilon_i)_{\vec{u}_i}, \text{ rate of change of storage}$$

$$(6.36b) \quad \frac{d(\varrho_i \bar{Q}_i)}{dt} \int_{\vec{u}_i} + \varrho_i \bar{Q}_i \operatorname{div} \vec{u}_i = \varrho_i \frac{d\bar{Q}_i}{dt} \int_{\vec{u}_i} + \varrho_i^e \bar{Q}_i = \varrho_i \bar{S}_i - \operatorname{div} \vec{T}_i \int_{\vec{u}_i} \\ = \operatorname{reyn}(\varrho_i \bar{Q}_i)_{\vec{u}_i}, \text{ rate of change of storage}$$

$$(6.37a) \quad \frac{d}{dt} \int_{\vec{u}} \varrho_i \varepsilon_i d^3v = \int_{\vec{u}} \varrho_i \mu_i d^3v - \oint_{\vec{u}} (d^2s \cdot \{ \vec{H}_i + \vec{j}_{di} \varepsilon_i \})_{\vec{u}}$$

$$(6.37b) \quad \frac{d}{dt} \int_{\vec{u}} \varrho_i \bar{Q}_i d^3v = \int_{\vec{u}} \varrho_i \bar{S}_i d^3v - \oint_{\vec{u}} \overline{d^2s \cdot \{ \vec{T}_i + \vec{j}_{di} \bar{Q}_i \}}_{\vec{u}}$$

$$(6.38a) \quad \operatorname{reyn}(\varrho_i \varepsilon_i)_{\vec{u}} = \varrho \frac{d(c_i \varepsilon_i)}{dt} \int_{\vec{u}} = \varrho_i \mu_i - \operatorname{div}(\vec{H}_i + \vec{j}_{di} \varepsilon_i)_{\vec{u}_i}$$

$$(6.38b) \quad \operatorname{reyn}(\varrho_i \bar{Q}_i)_{\vec{u}} = \varrho \frac{d(c_i \bar{Q}_i)}{dt} \int_{\vec{u}} = \varrho_i \bar{S}_i - \operatorname{div}(\vec{T}_i + \vec{j}_{di} \bar{Q}_i)_{\vec{u}_i}$$

$$(6.39a) \quad \frac{\partial}{\partial t} \int_V \varrho_i \varepsilon_i d^3v = \int_V \varrho_i \mu_i d^3v - \oint_S (d^2s \cdot \{ \vec{H}_i + \vec{j}_{di} \varepsilon_i + \vec{u} \varepsilon_i \varrho_i \})_{\vec{u}_i}$$

$$(6.39b) \quad \frac{\partial}{\partial t} \int_V \varrho_i \bar{Q}_i d^3v = \int_V \varrho_i \bar{S}_i d^3v - \oint_S \overline{d^2s \cdot \{ \vec{T}_i + \vec{j}_{di} \bar{Q}_i + \vec{u} \bar{Q}_i \varrho_i \}}_{\vec{u}_i}$$

$$(6.40a) \quad \operatorname{reyn}(\varrho_i \varepsilon_i)_{\vec{u}} = \frac{\partial(\varrho_i \varepsilon_i)}{\partial t} \int_{\vec{u}} = \varrho_i \mu_i - \operatorname{div}(\vec{H}_i + \vec{j}_{di} \varepsilon_i + \vec{u} \varepsilon_i \varrho_i)_{\vec{u}_i}$$

$$(6.40b) \quad \operatorname{reyn}(\varrho_i \bar{Q}_i)_{\vec{u}} = \frac{\partial(\varrho_i \bar{Q}_i)}{\partial t} \int_{\vec{u}} = \varrho_i \bar{S}_i - \operatorname{div}(\vec{T}_i + \vec{j}_{di} \bar{Q}_i + \vec{u} \bar{Q}_i \varrho_i)_{\vec{u}_i}$$

It is worth to notice that the simplest form of the left-hand side appears in the balances for substantial and immobile points, Eqs.(6.38) and (6.40), respectively. In the componential balances

(6.35) and (6.36), the right-hand side is concise, because they reveal only one kind of transfer, namely the non-diffusive transfer.

The above collection of 16 basic balances (from 6.25 to 6.40) consists of the scalar and vector groups. The theorem on equivalence of balances (Sec. 4.10) is applicable to either group of 8 equations.

Exercise 6.10. Derive the balances of i -th mass (6.14a,b), (6.15a,b), (6.16a,b), (6.17a,b) from the basic balances of CQ of the types (6.33a), (6.34a), (6.39a), (6.40a), (6.37a), (6.38a), (6.35a), (6.36a).

Instructions. Show with use of definitions (6.22), (6.23), (6.2), (6.1) and (6.3) that for the i -th mass we have $\varepsilon_i = 1$, $\mu_i = \varrho_i^6 / \rho_i$, and that the i -th mass is not subjected to non-diffusive transfer (Eq. 6.7a), or $\bar{H}_i = 0$. Use Eq. (6.9) in case of need.

Exercise 6.11. Derive the balances of i -th mass (6.14b), (6.15b), (6.16b) and (6.17b) from the transformation (6.26a).

Instructions as for Ex. 6.10.

6.8. The Componential Momentum. The Diffusive Stress

While the momentum of mixture composed of grains belonging to various species is the sum of momentums of all the grains, the momentum of i -th component is the sum of momentums of solely i -th grains. In this approach, the i -th momentum is a vector CQ of the storage partial density equal to the componential velocity \bar{u}_i , and the volume density $\varrho_i \bar{u}_i$ (see definitions 6.22 and 6.24b).

It has to be emphasized that the storage volume-density of i -th momentum is equal to the i -th mass-flux density through immobile surface (Eq. 6.7b). For that reason we need not even postulate that the sum of momentums of all the components is equal to the momentum of mixture; this is obtained from Eq. (6.8b) explicitly. From the classification (6.11) we conclude that not all the momentum of i -th

component but only its part of density $\rho_i \bar{u}$ contributes to the resultant momentum. The remaining "diffusive momentums" of densities \bar{j}_{di} become mutually cancelled when added (Eq.6.10), in the same way as the thermal momentums within a discrete set of grains (compare Eq.2.7c).

Substituting ~~$\bar{u}_i, \bar{A} = \rho_i \bar{u}_i$~~ $\bar{Q}_i = \bar{u}_i$ into Eqs.(6.26b) and (6.27b), we obtain the Reynoldsians of the i -th momentum density:

$$(6.41a) \quad \overrightarrow{\text{reyn}}_{\bar{u}_i}(\rho_i \bar{u}_i) = \underbrace{\rho \frac{d(c_i \bar{u}_i)}{dt} \bar{u} + \text{div} \bar{j}_{di} \bar{u}_i}_{\text{II form}} = \underbrace{\rho_i \frac{d\bar{u}_i}{dt} \bar{u}_i + \rho \xi_i \bar{u}_i}_{\text{IV form}}$$

$$(6.41b) \quad \overrightarrow{\text{reyn}}_{\bar{u}}(\rho_i \bar{u}_i) = \rho \frac{d(c_i \bar{u}_i)}{dt} \bar{u}$$

Substituting $\bar{w} = \bar{u}_i$, $\bar{A} = \rho_i \bar{u}_i$ into Eq.(3.30b) and using the IV form of Reynoldsian (6.41a), we have

$$(6.42a) \quad \underbrace{\frac{d}{dt} \int_{V_{\bar{u}_i}} \rho_i \bar{u}_i d^3v}_{\substack{\text{rate of change of} \\ \text{the } i\text{-th momentum} \\ \text{stored within } i\text{-th} \\ \text{componential region}}} = \underbrace{\int_{V_{\bar{u}_i}} \left\{ \rho_i \frac{d\bar{u}_i}{dt} \bar{u}_i \right\}}_{\substack{\text{influence of} \\ \text{change in the} \\ i\text{-th velocity}}} + \underbrace{\int_{V_{\bar{u}_i}} \left\{ \rho \xi_i \bar{u}_i \right\} d^3v}_{\substack{\text{influence of} \\ \text{change in the} \\ i\text{-th mass}}}$$

Contrary to the substantial region and the total mass, the i -th componential region does not contain a time-constant storage of i -th mass. Thus, the change of component's momentum is influenced not only by the change in velocity but also by the change in mass, manifested by its creation or annihilation.

The expression $\rho \xi_i \bar{u}_i$ has been called (Prigogine and Mazur, see Bibliography 10) the vector of conversion of momentum between the reacting components. This interpretation seems to be inappropriate from the viewpoint of balance axiom (4.1), because Eq.(6.42a) is not yet a balance, being a development of its left-hand side only. Even if we assumed the vector $\rho \xi_i \bar{u}_i$ to represent the conversion, the first term of the right-hand side of Eq.(6.42a) would certainly

not be the i -th momentum transfer through componential shell.

In order to compare the rates of change of the i -th momentum storage in the componential and substantial regions, we replace \bar{A} with $\varrho_i \bar{u}_i$ in Eq.(3.30b), make successive substitutions $\bar{w} = \bar{u}_i$ and $\bar{w} = \bar{u}$, then successively use the II form of Reynoldsian (6.41a) (which does not directly reveal the influence of i -th mass production) and the Reynoldsian (6.41b):

$$(6.42b) \quad \frac{d}{dt} \int_{\substack{V_{\bar{u}_i} \\ \rightarrow \bar{u}_i}} \varrho_i \bar{u}_i d^3V = \int_{\substack{V_{\bar{u}_i} \\ \rightarrow \bar{u}_i}} \left\{ \varrho \frac{d(c_i \bar{u}_i)}{dt} \rightarrow \bar{u} + \overline{\text{div}} \overline{\overline{\overline{j_{di} \bar{u}_i}}} \right\} d^3V \quad \left\{ \begin{array}{l} \text{within } i\text{-th} \\ \text{componential} \\ \text{region} \end{array} \right.$$

$$(6.42c) \quad \frac{d}{dt} \int_{\substack{V \\ \rightarrow \bar{u}}} \varrho_i \bar{u}_i d^3V = \int_{\substack{V \\ \rightarrow \bar{u}}} \varrho \frac{d(c_i \bar{u}_i)}{dt} \rightarrow \bar{u} d^3V \quad \left\{ \begin{array}{l} \text{within} \\ \text{substantial} \\ \text{region} \end{array} \right.$$

Comparing the right-hand sides, we find that at the instant when the regions $V_{\bar{u}_i}$ and $V_{\bar{u}}$ cover each other, the same storage of i -th momentum changes differently. After applying the GGO theorem (1.34a) the difference between Eqs.(6.42b) and (6.42c) assumes the shape of the surface integral of tensor $\overline{\overline{\overline{j_{di} \bar{u}_i}}}$, which has been called the i -th diffusive stress $\bar{\bar{\bar{\Pi}}}_{di}$ by Nachbar, Williams and Penner (see Bibliography 11). The expression for this quantity can be developed with the help of Eq.(6.9):

$$(6.43a) \quad \bar{\bar{\bar{\Pi}}}_{di} = \overline{\overline{\overline{j_{di} \bar{u}_i}}} = \varrho_i \overline{\overline{\overline{\bar{u}_i \bar{u}_i}}} - \varrho_i \overline{\overline{\overline{\bar{u} \bar{u}_i}}}$$

It results from Eq.(6.8b) that the diffusive stress for the mixture as a whole $\bar{\bar{\bar{\Pi}}}_d$ is a symmetric tensor composed of the dyadics of vectors \bar{u}_i and \bar{u} :

$$(6.43b) \quad \bar{\bar{\bar{\Pi}}}_d = \sum \bar{\bar{\bar{\Pi}}}_{di} = \sum \overline{\overline{\overline{j_{di} \bar{u}_i}}} = \sum \varrho_i \overline{\overline{\overline{\bar{u}_i \bar{u}_i}}} - \varrho \overline{\overline{\overline{\bar{u} \bar{u}}}}$$

Though the quantities $\bar{\bar{\bar{\Pi}}}_{di}$ and $\bar{\bar{\bar{\Pi}}}_d$ are of dimension of stress, they are of conventional character, in the same way as the Reynolds stress in the theory of turbulence. According to classification (6.31b) at

$\bar{Q}_i = \bar{u}_i$, the tensor $\bar{\Pi}_{di}$ (6.43a) describes the diffusive transfer of i -th momentum, or the capture due to the relative motion of componential surface with respect to the substantial surface.

Exercise 6.12. Prove the following transformation concerning the sum of storages of momentums of all the components within substantial region:

$$(6.44) \quad \frac{d}{dt} \int_{V_{\bar{u}}} \sum_i \rho_i \bar{u}_i d^3V = \int_{V_{\bar{u}}} \rho \frac{d\bar{u}}{dt} d^3V$$

Instruction. Apply Eq.(6.8b) and rule (5.12b) at $\bar{Q} = \bar{u}$.

6.9. The Balance of Componential Momentum

It is not obvious by any means whether Newton's Second Law of Motion, which is applicable to a single grain, can be extended over the set of all the i -th grains within i -th componential region $V_{\bar{u}_i}$. We are exposed to some doubts arising from the fact of creation and annihilation of the component mass, as well as from the difficulties in phenomenological determination of the intra-componential force interaction. We have to ask questions: are we allowed, from the phenomenological viewpoint, to regard the components of substance as a set of separate but mutually penetrating bodies?; how the force exerted on mixture as a whole should be decomposed among particular components?; how to describe the forces interacting solely among the components and vanishing when summed up over a closed set?

We shall not go into the intricate considerations on the phenomenological theory of multicomponent mixture, which is not too much advanced and consistent these days. Instead of it, we shall present a fairly sketchy form of the balance of componential momentum, deduced with the help of reasoning similar to that of Bearman and Kirkwood (see Bibliography 12).

At first we assume that the i -th component within i -th componential region is subjected to a ^{body} ~~bulk~~ force of the (volume-) density

$\bar{\mathbf{F}}_i$, and to a surface force expressed by the stress $\bar{\bar{\mathbf{T}}}_i$.

The componential bulk force results from exertions of the gravity and the long-range electromagnetic field, as well as from the resistance accompanying the diffusive motion. When summed up over the set of all the components, the diffusion-resistance forces become mutually cancelled, while the exertions mentioned give the bulk force of density $\bar{\mathbf{F}}$, acting on the mixture as a whole (compare Eq.5.22):

$$(6.45) \quad \sum \bar{\mathbf{F}}_i = \bar{\mathbf{F}}_g + \bar{\mathbf{F}}_{ed} = \bar{\mathbf{F}}$$

The componential stress $\bar{\bar{\mathbf{T}}}_i$ consists of the pressure part, expressed by the scalar componential pressure p_i , and of the viscous part $\bar{\bar{\mathbf{T}}}_{vi}$. In the state of local thermodynamic equilibrium, p_i is the product of the resultant pressure of the mixture p and the volume-fraction of i -th component (Dalton's law). In such circumstances the volume-density of the componential pressure force $-\overline{\text{grad}} p_i$ consists of two terms, the first of them expressing the diffusio-motive force (of the free diffusion). When summed up over the set of all the components, the first terms vanish, while the second terms give the volume-density of the pressure force $-\overline{\text{grad}} p$ (Eq. 5.27b), with respect to the mixture as a whole.

Now we postulate (being not absolutely positive!) that the balance of i -th momentum within i -th componential region obeys Newton's Second Law of Motion:

$$(6.46) \quad \frac{d}{dt} \int_{\bar{\mathbf{V}}_{\bar{\mathbf{u}}_i}} \rho_i \bar{\mathbf{u}}_i d^3v = \int_{\bar{\mathbf{V}}_{\bar{\mathbf{u}}_i}} \bar{\mathbf{F}}_i d^3v - \oint_{\bar{\mathbf{S}}_{\bar{\mathbf{u}}_i}} d^2\bar{\mathbf{s}} \cdot \bar{\bar{\mathbf{T}}}_i$$

Basing on the equivalence of Eqs.(6.35b) and (6.36b) at $\bar{\mathbf{Q}}_i = \bar{\mathbf{u}}_i$, $\rho_i \bar{\mathbf{S}}_i = \bar{\mathbf{F}}_i$, $\bar{\bar{\mathbf{T}}}_i = \bar{\bar{\mathbf{T}}}_i$, we obtain the balance of componential momentum at a componential point, or the equation of motion of i -th componen

$$(6.47a) \quad \overline{\text{reyn}}(\rho_i \bar{\mathbf{u}}_i) = \rho_i \frac{d\bar{\mathbf{u}}_i}{dt} \rightarrow \bar{\mathbf{u}}_i + \rho \zeta_i \bar{\mathbf{u}}_i = \bar{\mathbf{F}}_i - \text{div} \bar{\bar{\mathbf{T}}}_i$$

From the equivalence of Eqs.(6.36b) and (6.38b) with substitution (6.43a) we obtain the balance of componential momentum at a substantial point:

$$(6.47b) \quad \overrightarrow{\text{reyn}}(\varrho_i \bar{u}_i) = \varrho \frac{d(c_i \bar{u}_i)}{dt} \bar{u} = \bar{F}_i - \text{div}(\bar{\Pi}_i + \bar{\Pi}_{di})$$

Summing up the balances (6.47b) over the closed set (the substantial point being the same for all the components), and considering Eqs.(6.8c) and (6.45), we conclude that

$$(6.48) \quad \varrho \frac{d\bar{u}}{dt} \bar{u} = \bar{F} - \text{div} \sum (\bar{\Pi}_i + \bar{\Pi}_{di})$$

Comparing the result obtained with the balance (5.29b) and considering Eq.(6.43b), we find the relation between the resultant stress $\bar{\Pi}$ in the mixture and the componential stresses:

$$(6.49) \quad \bar{\Pi} = \sum (\bar{\Pi}_i + \bar{\Pi}_{di}) = \sum \bar{\Pi}_i + \bar{\Pi}_d$$

We see that the mixture's stress is not equal to the sum of all the componential stresses exerted on particular components within their own regions, The difference is the diffusive stress $\bar{\Pi}_d$.

Applying several simplifying assumptions (stationary state, the binary incompressible mixture at rest, local thermodynamic equilibrium, purely free diffusion), we are able to transform the component's momentum balance (6.47b) into such a form, in which the diffusion-resistance force is equilibrated with the diffuso-motive force, and which is, in fact, a transformed Fick's law (6.13).

6.10. Kinetic Energy of a Component and the Diffusion Energy

The sum of kinetic energies of the ordered motions of i-th grains is the kinetic energy of i-th component of substance, which can be regarded as a scalar Q_i .

According to definitions (6.22) and (6.24), with ε_i replaced with $(\bar{u}_i^2/2)$, the partial density and the volume-density of storage of the i-th componential kinetic energy are, respectively, equal to

$$(6.50a) \quad \epsilon_{ki} = (\bar{u}_i^2/2),$$

$$(6.50b) \quad \rho_i \epsilon_{ki} = \rho_i (\bar{u}_i^2/2)$$

We introduce the concept of the i-th componential diffusion energy, characterized by the following partial density and volume-density of storage:

$$(6.51a) \quad \epsilon_{di} = \frac{1}{2}(\bar{j}_{di}^2)/\rho_i^2,$$

$$(6.51b) \quad \rho_i \epsilon_{di} = \frac{1}{2}(\bar{j}_{di}^2)/\rho_i$$

The sum of the componential diffusion energies over all the components is called the mixture's diffusion energy. According to definitions (5.7), (5.9a), and Eq.(6.3), the storage of the discussed scalar SQ is characterized by the following mass-density and volume-density:

$$(6.52a) \quad \epsilon_d = \sum c_i \epsilon_{di},$$

$$(6.52b) \quad \rho \epsilon_d = \sum \rho_i \epsilon_{di}$$

Applying substitution (6.9) into Eq.(6.50b) and using the denotation (6.51b), we have

$$(6.53) \quad \rho_i \epsilon_{ki} = \rho_i \left(\frac{\bar{u}_i^2}{2} \right) + (\bar{u}_i \bar{j}_{di}) + \rho_i \epsilon_{di}$$

We sum up the volume-densities of the componential kinetic energies over all the components. Taking into consideration Eqs.(6.4a), (6.10) and (6.52b), we obtain:

$$(6.54) \quad \underbrace{\sum \rho_i \epsilon_{ki}}_{\substack{\text{the sum of} \\ \text{components'} \\ \text{kinetic energies}}} = \underbrace{\rho \left(\frac{\bar{u}^2}{2} \right)}_{\substack{\text{mixture's} \\ \text{kinetic energy}}} + \underbrace{\rho \epsilon_d}_{\substack{\text{mixture's} \\ \text{diffusion energy}}}$$

We have found that the sum of kinetic energies of all the components is not equal to the kinetic energy of mixture, but greater by the diffusion energy. The reason of this inequality is that the sum of componential kinetic energies concerns all the ordered motions, while the mixture's kinetic energy is due to only those ordered motions that give non-zero macroscopic momentum. The diffusion energy results from the ordered motions with mutually cancelling momentums

Since they are motions of thermal nature, the diffusion energy, from the phenomenological viewpoint, is regarded as a part of thermal energy, in the same way as the energy of the disordered grain motions is treated (compare Eq.2.8b).

In the resting mixture, having zero-valued kinetic energy as a whole, the sum of componential kinetic energies is equal to the diffusion energy. On the other hand, at a fairly intense flow of mixture and relatively slow process of diffusion, the diffusion energy becomes negligible, so we can simplify Eq.(6.54) as follows:

$$(6.55) \quad \sum \varrho_i \varepsilon_{ki} \cong \varrho \left(\frac{\bar{u}^2}{2} \right) \quad \text{at slow diffusion}$$

It is possible to derive (Ex.6.13) the balance of i-th kinetic energy within i-th componential region in the following form:

(6.56)

$$\underbrace{\frac{d}{dt} \int_{V_{\bar{u}_i}} \varrho_i \varepsilon_{ki} d^3V}_{\text{rate of change of storage}} = \underbrace{\int_{V_{\bar{u}_i}} \left\{ -\varrho_i^e \varepsilon_{ki} + (\bar{F}_i \bar{u}_i) + (\bar{P}_i \bar{g} \bar{r} \bar{a} \bar{d} \bar{u}_i) \right\} d^3V}_{\text{production}} - \underbrace{\oint_{S_{\bar{u}_i}} (d^2\bar{s} \bar{P}_i \bar{u}_i)}_{\text{non-diffusive transfer}}$$

The balance given displays some analogies with the kinetic energy balance for one-component fluid. The latter balance, however, has none production term corresponding to $-\varrho_i^e \varepsilon_{ki}$, expressing the creation of i-th kinetic energy when the i-th mass is annihilated, and vice versa. Since the balance (6.56) comprises the component's contributions to both kinetic energy of mixture and internal energy (manifested as the diffusion energy) in a composite way, its phenomenological interpretation is difficult.

Applying the theorem on equivalence of balances (6.35a) and (6.38a), we can transform Eq.(6.56) into the balance at substantial point, then sum up the balances of componential kinetic energies over all the components of mixture. According to Eq.(6.54), the

obtained sum minus the kinetic energy balance for mixture as a whole will result in the balance of the diffusion energy of mixture. This can be achieved at the cost of very tedious labour, requiring many substitutions and transformations. The obtained balance contains 9 terms of difficult and partly ambiguous phenomenological interpretation.

It is sure, however, that the diffusion energy is subjected to conversion with the electromagnetic energy and various sub-categories of the internal energy, while its transfer is of both diffusive and non-diffusive character. It is possible to prove that the flow of electric current is accompanied with considerable creation and annihilation of the diffusion energy, though its storage usually is negligibly small. For example in the electrolyte of an ordinary car battery being charged in usual conditions, the equivalent of storage of the diffusion energy is created within time of the order of 10^{-14} second; at the same time this energy is annihilated for the benefit of energy of disordered motions, manifested as Joule's heat.

Exercise 6.13. Derive the balance (6.56).

Instructions. Multiply the balance (6.47a) by \bar{u}_i in a scalar product, apply transformation (1.30b) and denotation (6.50a). Rearrange the left-hand side so that it assumes the IV form of Reynoldsonian (6.26a). Hence we obtain the balance of i -th kinetic energy at the componential point:

$$(6.57) \quad \text{reyn}(\rho_i \varepsilon_{ki}) \underset{\rightarrow \bar{u}_i}{=} - \rho_i \zeta_i \varepsilon_{ki} + (\bar{F}_i \bar{u}_i) + (\bar{\Pi}_i \bar{g} \bar{r} \bar{a} \bar{d} \bar{u}_i) - \text{div } \bar{\Pi}'_i \bar{u}_i$$

Considering the equivalence of Eqs. (6.36a) and (6.35a), we obtain the balance (6.56).

6.11. The Internal Energy and Enthalpy of a Component

Because of the complexity of intra-molecular bonds, it is very difficult to define the concept of componential internal energy.

While the thermal motion energy can be allotted among the molecules of different species, it is not clear how to distribute the field energy of short-range interaction among the thoroughly mixed grains of various components. This is the reason why we risk arbitrariness and indeterminacy when asserting that the i -th component contains some i -th internal energy (this uncertainty concerns mainly the liquid).

Nevertheless, in the phenomenological approach, the i -th internal energy is regarded as a scalar CQ of the storage partial density ϵ_{Ii} and volume-density $\varrho_i \epsilon_{Ii}$ (see definitions 6.22, 6.24a). The enthalpy of i -th component is also a scalar CQ, characterized by the partial density of storage H_i :

$$(6.58) \quad H_i = \epsilon_{Ii} + p v_{\bullet i}$$

where $v_{\bullet i}$ is the partial specific volume, or, rigorously, the partial density of volume (the volume occupied by i -th component per unit of i -th mass).

The distribution of mixture's internal energy among the components must be consistent with the ways of distribution of other energies of substantial character, which are the gravitational and kinetic energies.

Since each species of grain is subjected to identical gravitational potential Γ , the sum of gravitational energies of all the components is equal to the mixture's gravitational energy (see Sec. 5.11 and Eq. 6.4a):

$$(6.59) \quad \sum \varrho_i \Gamma = \varrho \Gamma$$

Each of the components is the site of three kinds of substantial energies of total density $\varrho_i(\Gamma + \epsilon_{ki} + \epsilon_{Ii})$. If we sum up the given expression over all the components, we have to obtain the density of sum of the gravitational, kinetic and internal energy, with respect to the mixture as a whole:

$$(6.60) \quad \sum \varrho_i (\Gamma + \varepsilon_{ki} + \varepsilon_{Ii}) = \varrho \left\{ \Gamma + (\bar{u}^2/2) + \varepsilon_I \right\}$$

Taking Eqs.(6.59) and (6.54) into consideration, we have

$$(6.61) \quad \varrho \varepsilon_I = \sum \varrho_i \varepsilon_{Ii} + \varrho \varepsilon_d$$

We have found that the internal energy of mixture is not equal to the sum of internal energies of all the components, but greater by the diffusion energy. This is ~~an~~ obvious consequence of relation (6.54) commented in Sec. 6.10.

Though the storage of the diffusion energy usually is negligibly small as compared to the storages of other energies, nevertheless its conversion and transfer can exert a strong influence on the balance of internal energy. Leaving out tedious mathematical deduction, we shall outline the reasoning in short.

It results from Eq.(6.61) that by the addition of the diffusion energy balance to the sum of balances of componential internal energies, we obtain the balance of mixture's internal energy. Certain assumptions with respect to conversion and transfer of the componential internal energies, and full development of the diffusion energy balance make it possible to reveal that the diffusive transfer of internal energy (see Eq.5.60) is influenced by the so-called intra-componential pressure. Owing to this phenomenon and according to Eq.(6.58), the vector \bar{J}_d can be expressed as follows:

$$(6.62) \quad \bar{J}_d = \sum H_i \bar{J}_{di} = \sum \varepsilon_{Ii} \bar{J}_{di} + p \sum v_{oi} \bar{J}_{di}$$

It seems astonishing indeed that the diffusive transfer of internal energy depends ultimately on the carried componential enthalpies. The classification (6.31a) suggests rather that \bar{J}_d is a function of componential internal energies, being equal to $\sum \varepsilon_{Ii} \bar{J}_{di}$. Such a result would be obtained if we ignored the share of the diffusion energy within the internal energy, assuming that $\varrho \varepsilon_I = \sum \varrho_i \varepsilon_{Ii}$.

This way of reasoning had been commonly practised until Truesdell (see Bibliography 7) formulated Eq.(6.61) in 1957. Save for the particular case of mixture of incompressible components, the neglect of the diffusion energy implies grave errors, because, in certain conditions, the difference $\sum (H_i - \varepsilon_{ii}) \bar{J}_{di}$ can be of the same order as that of \bar{J}_{tc} and \bar{J}_{tr} , which characterize the conductive and radiative transfer, respectively.

In the nonequilibrium thermodynamic, the non-convective transfer-flux density of internal energy (see Eq.5.60) is usually presented in two alternative forms

$$(6.63) \quad \bar{J}_{nc} = \bar{J}'_q + \sum \varepsilon_{ii} \bar{J}_{di} = \bar{J}''_q + \sum H_i \bar{J}_{di} ,$$

\bar{J}'_q and \bar{J}''_q being called the first-law and second-law heat flux, respectively.

In view of our previous considerations, the flux \bar{J}'_q is a purely conventional quantity, deprived of interpretation, because $\sum \varepsilon_{ii} \bar{J}_{di}$ describes, generally speaking, only a part of the diffusive transfer of internal energy.

On the other hand, comparing Eqs.(6.63), (5.60) and (6.62), we conclude that the flux $\bar{J}''_q = \bar{J}_{tc} + \bar{J}_{tr}$ virtually is the entire conductive-radiative transfer of internal energy, and for that reason it has a definite physical meaning.

The use of substitutions (6.62) and (5.60) makes possible to re-write the balances of internal energy (5.58a) and enthalpy (5.67) in more specified forms:

$$(6.64) \quad \rho \frac{d\varepsilon_I}{dt} \rightarrow \bar{u} = G_{pm} + \left(\frac{\bar{E}}{\rightarrow \bar{u}} \frac{\bar{I}}{\rightarrow \bar{u}} \right) - \left(\rho p \frac{dv}{dt} \rightarrow \bar{u} - (\bar{\Pi}_v \bar{g} \bar{r} \bar{a} \bar{d} \bar{u}) \right) + \\ - \text{div}(\bar{J}_{tr} + \bar{J}_{tc} + \sum H_i \bar{J}_{di})$$

$$(6.65) \quad \rho \frac{dH}{dt} \rightarrow \bar{u} = G_{pm} + \left(\frac{\bar{E}}{\rightarrow \bar{u}} \frac{\bar{I}}{\rightarrow \bar{u}} \right) + \frac{dp}{dt} \rightarrow \bar{u} - (\bar{\Pi}_v \bar{g} \bar{r} \bar{a} \bar{d} \bar{u}) + \\ - \text{div}(\bar{J}_{tr} + \bar{J}_{tc} + \sum H_i \bar{J}_{di})$$

It is worth to notice that the diffusive transfer of both internal energy and enthalpy of the mixture (the sum terms in Eqs. 6.64 and 6.65) lies in the carrying of the componential enthalpies in their diffusive motion.

7. The Balances of Total Energy and Entropy

7.1. The Balance of Total Energy

From the phenomenological viewpoint, the total energy forms closed set of 4 specific energies (kinetic, gravitational, internal, and electromagnetic, Sec. 2.7). In order to obtain the balance of total energy, we have only to add the balances of specific energies within the same region or at the same point. On account of the non-substantial character of electromagnetic energy, the most simple reference object of balancing is an immobile point. The previously obtained balances (5.42b) (with substitution 5.22), (5.44b), (5.57b) (with substitution 5.60), and (4.49) have been tabulated in form of the set Eqs.(7.1).

Since the kinetic, gravitational, and internal energies are all scalar SQs, the sum of them we shall call the substantial energy of mass density

$$(7.2a) \quad \epsilon_s = (\bar{u}^2/2) + \Gamma + \epsilon_I$$

The total energy, as the sum of substantial and non-substantial (electromagnetic) energies, is a scalar EQ of volume-density

$$(7.2b) \quad W = W_{em} + \varrho \epsilon_s$$

Considering the divergence terms in Eqs.(7.1), we must remember that in case of electromagnetic energy as a non-substantial quantity, the decomposition of transfer into the convective and non-convective parts would be meaningless. This energy is not carried with substance, though it can be captured through a moving surface, in the same way as any other EQ.

(7.1) Balances of specific energies at immobile point

Category of energy	Rate of change of storage	Production	Transfer	
			non-convective	convective
substantial	kinetic	$\frac{\partial}{\partial t} \left\{ \rho \left(\frac{\bar{u}^2}{2} \right) \right\} = (\bar{F}_g \bar{u}) + (\bar{F}_{ed} \bar{u}) + (\bar{\Pi} \bar{g} \bar{\nabla} \bar{u})$	$-\text{div } \bar{\Pi} \bar{u}$	$-\text{div} \left\{ \rho \left(\frac{\bar{u}^2}{2} \right) \bar{u} \right\}$
	gravitational	$= - (\bar{F}_g \bar{u})$		$-\text{div} \{ \rho \bar{\Gamma} \bar{u} \}$
	internal	$= G_{pm} + (\bar{E}_u \bar{I}_u) - (\bar{\Pi} \bar{g} \bar{\nabla} \bar{u})$	$-\text{div} (\bar{J}_{tr} + \bar{J}_{tc} + \bar{J}_d)$	$-\text{div} \{ \rho \varepsilon_I \bar{u} \}$
non-substantial			through immobile surface	
	electromagnetic	$\frac{\partial}{\partial t} W_{em} = - G_{pm} - (\bar{E} \bar{I}_u) - (\bar{F}_{ed} \bar{u})$	$-\text{div} (\bar{J}_{er} + \bar{J}_{ec})$	

Before we add the balances (7.1), it is convenient to assemble the non-convective transfer of substantial energy with the transfer of non-substantial electromagnetic energy through immobile surface. The obtained sum will be called the paraconvective transfer of total energy, its flux density being equal to

$$(7.3a) \quad \bar{J}_{pc} = \overline{\Pi^c \bar{u}} + \bar{J}_d + \bar{J}_{ec} + \bar{J}_{tc} + \bar{J}_{er} + \bar{J}_{tr}$$

Since the paraconvective transfer (appearing "beside the convection") does concern neither convection nor capture, none of its terms depends on the storage density of any specific energy.

The transfer of total energy through immobile surface is the sum of the paraconvective transfer of total energy and the convective transfer of substantial energy, its flux density being equal to

$$(7.3b) \quad \bar{J} = \bar{J}_{pc} + \varrho \epsilon_s \bar{u}$$

Summing up 4 balances of specific energies shown in Eqs.(7.1), we find that all the production terms cancel each other in pairs. Substituting Eqs.(7.2a,b) and (7.3a,b), we obtain a concise form of the balance of total energy at immobile point:

$$(7.4) \quad \frac{\partial W}{\partial t} = - \operatorname{div} \bar{J}$$

It is possible to show (Ex. 7.1) that the balance of total energy at substantial point is as follows:

$$(7.5) \quad \operatorname{reyn}_{\bar{u}} W = - \operatorname{div}(\bar{J}_{pc} - W_{em} \bar{u})$$

where $W_{em} \bar{u}$ is the capture-flux density of electromagnetic energy.

The balances (7.4) and (7.5) show that the only possible cause of change of storage of total energy, as a conservative quantity, is the transfer.

Exercise 7.1. Derive the balance (7.5).

Instructions. Use the property of equivalence of basic balances

(4.31a) and (4.30a). Consider the rule (4.20a) with substitutions $\bar{w} = \bar{u}$, $\varphi = w$, $\psi = 0$, $\bar{H} = \bar{J}$, then use Eqs.(7.3b) and (7.2b).

7.2. Classification of Conversions and Transfers of Energy

Fig. 7.1

The block diagram shown in Fig. 7.1 presents a synthesis of the energetic processes occurring in the substantial region. The contained total energy consists of 4 specific energies, which can be converted exclusively within this set at 4 possible combinations. While the production of total energy is none, the transfer between the substantial region and its surroundings can be realized by means of 7 distinct phenomena, in which, however, only 3 specific energies can participate.

The gravitational energy is subjected to conversion with the kinetic energy only. This is ^areversible conversion, customarily known as the work of gravitational force. The remaining 3 energies form a "conversion triangle": each of them is subjected to conversion with two others.

The conversion between the kinetic and electromagnetic energies is of reversible character, being customarily called the work of electrodynamic force. The conversion between the kinetic and internal energies (corresponding to "thermodynamic work") is partly reversible and irreversible. The same property characterizes the conversion between the electromagnetic and internal energies. All the irreversible conversions are manifested by creation of internal energy.

The technology takes advantage of the discussed processes. The principles of functioning of a few typical devices utilizing energy conversion are given below (the arrows indicate the direction from annihilation of one specific energy to creation of the other):

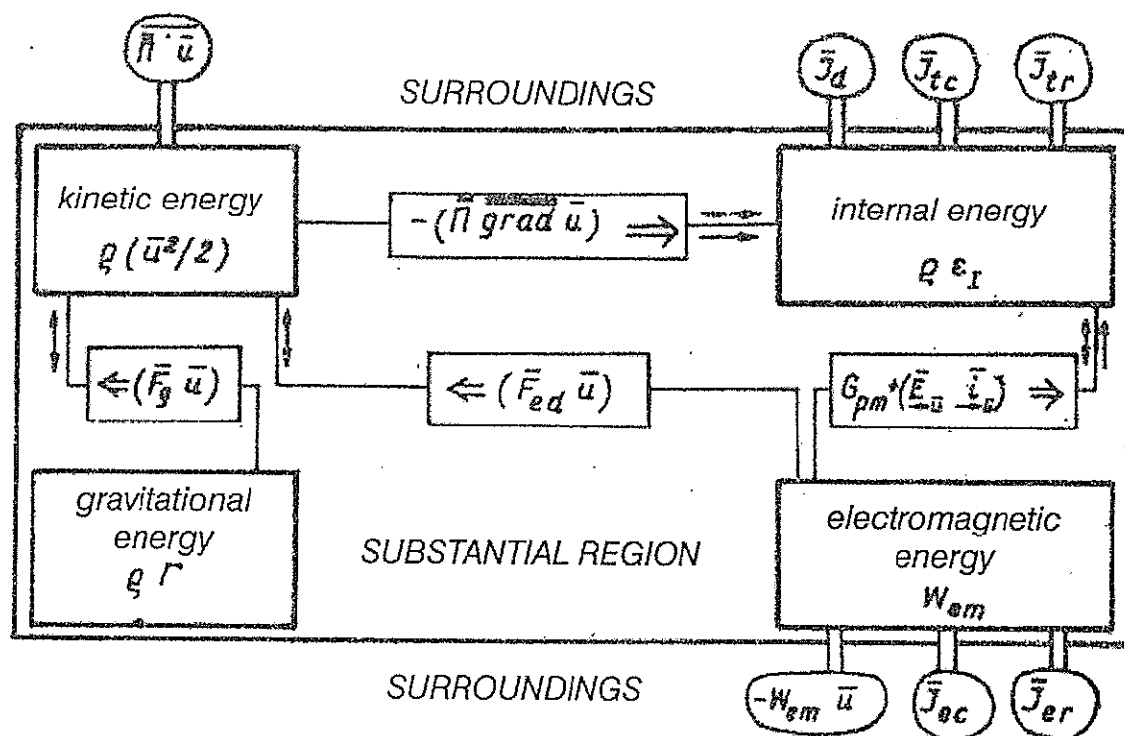
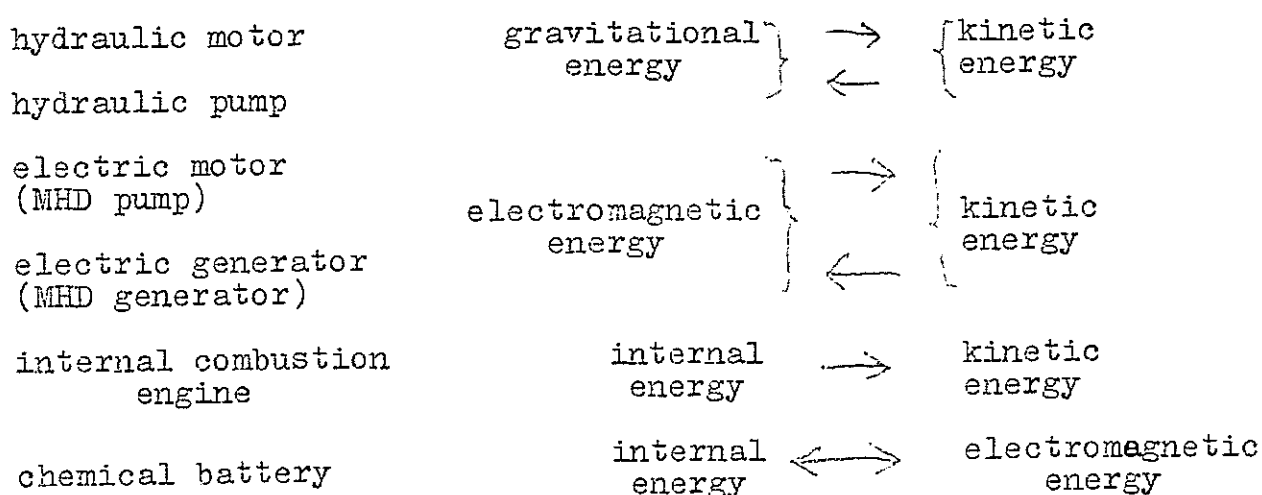


Fig. 7.1. Scheme of conversions (thin rectangles) and exchanges (thin ovals) of particular types of energies (thick rectangles). In conversions production densities of component energies (wide arrows) are given. Reversibility and irreversibility are marked by thin arrow with one or two arrow heads, respectively.



It is worth to notice that in the internal combustion engine, the conversion between two sub-categories of internal energy occurs as well, the chemical bond energy being annihilated for the benefit of thermal energy. The latter form of internal energy undergoes conversion into the kinetic energy.

The paraconvective transfer \bar{J}_{pc} (Eq.7.3a) consists of:

- (1) $\overline{\bar{\Pi} \cdot \bar{u}}$ non-convective transfer of kinetic energy due to motion of substance subjected to the fields of elastic, pressure, and viscous stresses (the stress transfer),
- (2) \bar{J}_d non-convective transfer of internal energy due to diffusion of the component masses of mixture (the diffusive transfer)
- (3) \bar{J}_{ec} conductive transfer of electromagnetic energy due to conduction of electric current, going together with the diffusion of carriers of charge (hence similarity between \bar{J}_d and \bar{J}_{ec}),
- (4) \bar{J}_{tc} conductive transfer of internal energy due to thermal conduction (some similarity between \bar{J}_{ec} and \bar{J}_{tc} results from the microphysical character of both processes),
- (5) \bar{J}_{er} radiative transfer of electromagnetic energy, going together with radiation of radio waves (of macroscopic wavelength)
- (6) \bar{J}_{tr} radiative transfer of internal energy, going together with thermal radiation (of microscopic wavelength).

The transfers \bar{J}_{er} and \bar{J}_{tr} have a common property: both are nothing else than electromagnetic radiation, differing in the ranges of frequency spectrum only. In contrary to the kinds of transfer listed as items (1) to (4), the radiative transfer (5) and (6) does not require the existence of substance to be realized.

For the substantial region (compare Eq.7.5), the paraconvective transfer is complemented with:

(7) $W_{em}\bar{u}$ the capture of electromagnetic energy due to motion of substantial surface.

This kind of transfer theoretically accompanies, for example, the plasma flow in non-homogenous electromagnetic field, though it usually is of negligible importance.

The discussed ways of transfer are principles of functioning of various devices transmitting the energy from one system to another, such as the drive-mechanisms and hydraulic/pneumatic servo-mechanism (with \bar{u} flux density utilized), the heat exchangers (\bar{J}_{tc} and \bar{J}_{tr}), electric networks (\bar{J}_{ec}), the radio-communication systems (\bar{J}_{er}), etc.

7.3. Thermodynamic Balance of Energy

We deal very often with energetic processes unaffected by electromagnetic phenomena. In such circumstances, since the terms G_{pm} , $(\bar{E}\bar{i}_{\rightarrow\bar{u}})$, $(\bar{F}_{ed}\bar{u})$, \bar{J}_{er} and \bar{J}_{ec} disappear ⁱⁿ ~~from~~ Eqs.(7.1), the total energy can be identified with the substantial energy, and Eq.(7.2b) assumes simplified form $W = Q\mathcal{E}_s$.

In the absence of electromagnetic phenomena, the total energy balance is called the thermodynamic balance of energy and can be formulated in 4 following versions (Ex. 7.2):

$$(7.6a) \quad \frac{d}{dt} \int_{\underline{V}_{\underline{u}}} \rho \varepsilon_s d^3v = - \oint_{\underline{S}_{\underline{u}}} (\bar{J}'_{pc} d^2\bar{s}), \quad (7.6b) \quad \rho \frac{d\varepsilon_s}{dt} \rightarrow \bar{u} = - \operatorname{div} \bar{J}'_{pc}$$

$$(7.7a) \quad \frac{\partial}{\partial t} \int_V \rho \varepsilon_s d^3v = - \oint_S (\{ \bar{J}'_{pc} + \rho \varepsilon_s \bar{u} \} d^2\bar{s}), \quad (7.7b) \quad \frac{\partial(\rho \varepsilon_s)}{\partial t} = - \operatorname{div}(\bar{J}'_{pc} + \rho \varepsilon_s \bar{u})$$

$$\text{at: } \varepsilon_s = (\bar{u}^2/2) + \Gamma + \varepsilon_I, \quad (7.8) \quad \bar{J}'_{pc} = \overline{\bar{n}' \bar{u}} + \bar{J}_d + \bar{J}_{tc} + \bar{J}_{tr}$$

thermodynamic balance of energy
(electromagnetic phenomena not involved)

The balance (7.6a) concerns the substantial region, called the closed system in the thermodynamic terminology. Its particular case is the adiabatic system, the boundary of which cannot be penetrated by the substance components in their diffusive motions ($\bar{J}_d = 0$) and allows the energy transfer by means of neither heat conduction nor thermal radiation ($\bar{J}_{tc}, \bar{J}_{tr} = 0$). Taking all those properties into account when considering Eqs.(7.8) and (7.6a), we conclude that the change of the total energy (excluding the electromagnetic energy) stored within an adiabatic system is equal to the work of surface forces (5.39b):

$$(7.9) \quad \frac{d}{dt} \int_{\underline{V}_{\underline{u}}} \rho \varepsilon_s d^3v = - \oint_{\underline{S}_{\underline{u}}} (\overline{\bar{n}' \bar{u}} d^2\bar{s}) \quad \text{thermodynamic balance of energy in adiabatic system}$$

Exercise 7.2. Derive the balances (7.6a,b) and (7.7a,b).

Instructions. Having disregarded electromagnetic phenomena, we substitute $W = \rho \varepsilon_s$ and $\bar{J}_{pc} = \bar{J}'_{pc}$ (Eq.7.8) into Eqs.(7.3b) and (7.4) in order to obtain the balance (7.7b). The remaining versions of balance result from equivalence of the basic balances (5.21a), (5.20a), (5.18a) and (5.19a).

7.4. The Balance of Dynamic Enthalpy

In the thermodynamic of flow we deal with a very convenient concept of the dynamic enthalpy (customarily called the stagnation

enthalpy). This is a scalar SQ of the same dimension as energy, and consisting of the (static) enthalpy (Eq.5.65a), kinetic energy and gravitational energy. The dynamic enthalpy is characterized by the storage mass-density H' and volume-density $\varrho H'$:

$$(7.10a) \quad H' = H + (\bar{u}^2/2) + \Gamma = (\bar{u}^2/2) + \Gamma + \varepsilon_I + pv = \varepsilon_S + pv$$

$$(7.10b) \quad \varrho H' = \varrho H + \varrho(\bar{u}^2/2) + \varrho\Gamma = \varrho(\bar{u}^2/2) + \varrho\Gamma + \varrho\varepsilon_I + p = \varrho\varepsilon_I + p$$

(alternative forms have been obtained with the help of Eq.7.2a).

The dynamic enthalpy is, of course, neither the total energy of the fluids, nor its substantial energy.

It is possible to show (Ex. 7.3) that the balances of dynamic enthalpy at substantial and immobile points are as follows:

$$(7.11a) \quad \varrho \frac{dH'}{dt} \rightarrow \bar{u} = \frac{\partial p}{\partial t} - \text{div}(\bar{\Pi}_V' \bar{u} + \bar{J}_d + \bar{J}_{tc} + \bar{J}_{tr})$$

$$(7.11b) \quad \frac{\partial(\varrho H')}{\partial t} = \frac{\partial p}{\partial t} - \text{div}(\bar{\Pi}_V' \bar{u} + \bar{J}_d + \bar{J}_{tc} + \bar{J}_{tr} + \varrho H' \bar{u})$$

(electromagnetic phenomena not involved)

In the stationary flow ($\partial/\partial t = 0$) simultaneously being of isentropic character, we leave out of consideration the phenomena of viscosity, diffusion, thermal conduction and radiation ($\bar{\Pi}_V', \bar{J}_d, \bar{J}_{tc}, \bar{J}_{tr} = 0$), as well as all the electromagnetic exertions. In such circumstances, Eq.(7.11b) becomes simplified:

$$(7.12a) \quad \text{div}(\varrho H' \bar{u}) = 0 \quad \text{for stationary isentropic flow}$$

It is possible to show (Ex. 7.4) that the above formula leads to the following equation:

$$(7.12b) \quad H' = H + (\bar{u}^2/2) + \Gamma = \text{const} \quad \text{along the streamline} \\ \text{for stationary isentropic flow}$$

In case of the perfect gas, the (static) enthalpy $H = c_p T$ depends on the isobaric specific heat c_p and temperature T , and the changes of gravitational energy can usually be disregarded, and this way Eq.(7.12b) assumes the following form:

$$(7.13) \quad H' = c_p T + (\bar{u}^2/2) = \text{const} \quad \text{along the streamline}$$

for stationary isentropic flow of perfect gas

Exercise 7.3. Derive the balances (7.11a,b).

Instructions. We consider Eq.(5.4c) and rules (3.3a) and (1.30a) in order to transform Eq.(5.66) into the form:

$$(7.14) \quad \varrho \frac{d(p\bar{v})}{dt} \bar{u} = \frac{dp}{dt} \bar{u} + p \operatorname{div} \bar{u} = \frac{\partial p}{\partial t} + \operatorname{div}(p\bar{u})$$

Having added this equation to the thermodynamic balance (7.6b) (electromagnetic phenomena excluded), we develop the vector \bar{J}'_{pc} (Eq.7.8) and apply substitutions (5.52b) and (7.10a). Having obtained Eq.(7.11a), we find Eq.(7.11b) from the equivalence of basic balances (5.19a) and (5.21a).

Exercise 7.4. Derive Eq.(7.12b).

Instructions. According to rule (1.30a), the left-hand side of Eq.(7.12a) is equal to $(\varrho \bar{u} \operatorname{grad} H') + H' \operatorname{div}(\varrho \bar{u})$, the latter term being zero because of the assumed stationary flow (Eq.5.5a), hence $(\bar{u} \operatorname{grad} H') = 0$. Next we use the same way of reasoning as in case of the Bernoulli equation (5.55), substituting Eq.(7.10a).

Exercise 7.5. Show that the hydraulic equation of Bernoulli (5.55) is a particular case of Eq.(7.12b).

Instructions. From the balance (5.58b) for stationary ($\partial/\partial t=0$) incompressible ($v = \text{const}$) and inviscid ($\bar{\Pi}_v = 0$) flow, being subjected to isentropic process ($G_{pm}, \bar{I}_{\vec{u}}, \bar{J}_{nc} = 0$), we obtain the relation $\operatorname{div}(\varrho \bar{\epsilon}_I \bar{u}) = 0$, resolving itself into $(\bar{u} \operatorname{grad} \bar{\epsilon}_I) = 0$ (see rule 1.30a and Eq.5.5a), so we find that $\bar{\epsilon}_I = \text{const}$ along the streamline. Considering this in Eq.(7.12b), substituting Eqs.(5.65a and (5.24), and setting $v = 1/\varrho$, we finally obtain Eq.(5.55).

7.5. The Energetic Balance of an Electro-Roto-Flow System

In case of the stationary flow ($\partial/\partial t = 0$), the balance of total energy at immobile point (7.4) becomes simple equation $\operatorname{div} \bar{J} = 0$.

Integrating this expression within an immobile region, applying the GGO theorem (1.34a) and substitutions (7.3b) and (7.3a), and considering that in the stationary state $\bar{J}_{er} = 0$ (Eq.4.47b), we have

$$(7.15) \quad \oint_S \left(\bar{n} \bar{u} + \bar{J}_d + \bar{J}_{tr} + \bar{J}_{tc} + \bar{J}_{ec} + \varphi \epsilon_s \bar{u} \right) d^2s = 0$$

for stationary process in immobile region

In the described circumstances, the net transfer of total energy remains zero, because as much energy flows into the region as flows out. Of course, as a result of conversions occurring within the region, the fractional proportions among various energies entering and leaving need not be preserved. In the discussed problem, however, we shall consider only the transfer of particular energies flown through the region's shell, irrespective of all what ~~it~~ takes place in its interior.

The relation (7.15) is of basic importance for the theory of stationary-working electro-roto-flow systems, especially for the formulation of energetic balances and defining the efficiencies.

Fig. 7.2. The universal model of an electro-roto-flow system is shown in Fig. 7.2. It has a rigid casing, interior of which can contain one or several rotors revolving with constant angular velocity, as well as one or several open or closed regions filled with fluid (liquid or gas). The open region communicating with the surroundings forms a flow channel. Moreover, the system can be connected with the electric network enabling the flow of current either through solid wiring (of the rotor and stator), or across the fluid moving along the channel.

All three listed features (rotor, channel, electric circuit) need not be contained together in a particular device. The pure flow systems (e.g. the nozzles, heat exchangers) are deprived of the rotor and electric circuit. The electro-flow systems (e.g. the electric flow-heaters, MHD generators and pumps) do not contain

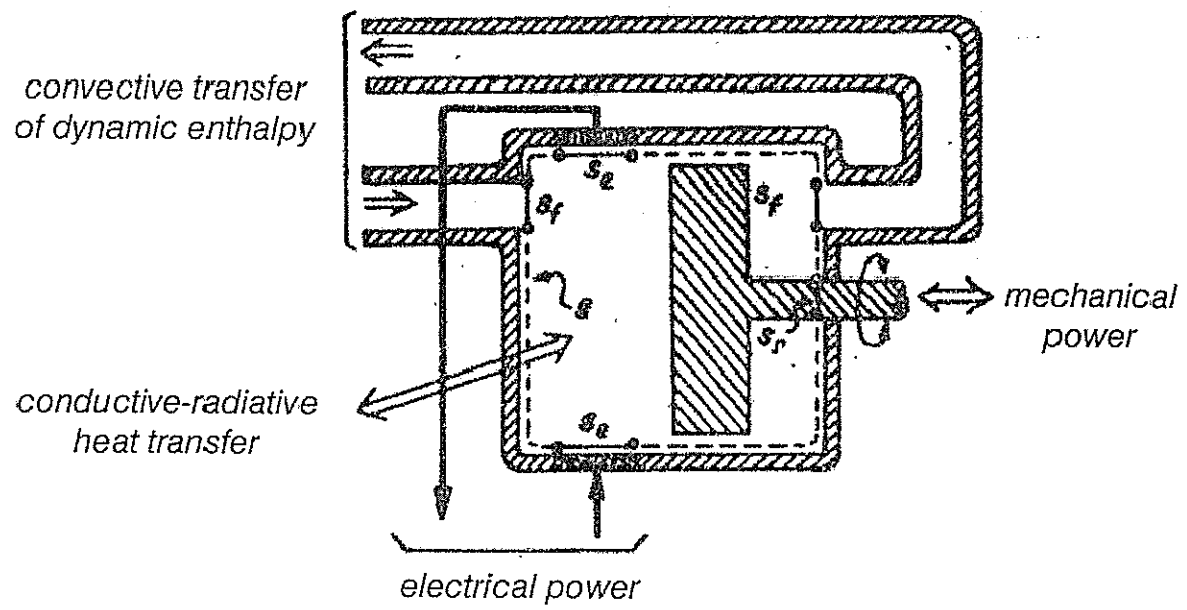


Fig. 7.2. Universal model of the electro-rotor-flow system.

rotors. The conventional electric machines (generators and motors) are electro-roto systems without flow channels, whereas the conventional flow machines (turbines, rotary compressors and pumps) are non-electric roto-flow systems.

As long as the rigid casing of the system remains at rest, or is in a state of non-accelerated motion in the terrestrial conditions, its physical inside surface (complemented with imaginary surfaces which cover the inlets and outlets of flow channels, and the cross-sections of the outer ends of rotor-shafts) can be regarded as the shell s of immobile balance region (Fig. 7.2). As it will prove, some of the terms being integrated over s in Eq.(7.15) assume non-zero values only on particular parts of the shell.

Notice that the product $(\bar{u} d^2\bar{s}) \neq 0$ solely on the surfaces covering the inlets and outlets of flow channels (surfaces denoted as $\sum s_f$). On the remaining parts of the shell s (the inside walls of the casing, the cross-sections of rotor-shafts) either $\bar{u} = 0$, or $\bar{u} \perp d^2\bar{s}$.

On the cross-sections of rotor-shafts (denoted as $\sum s_r$), the stress is of purely elastic character, $\bar{\Pi} = \bar{\Pi}_e$.

Notice that the product $(\bar{J}_{ec} d^2\bar{s}) = (\Psi \bar{I} d^2\bar{s}) \neq 0$ (compare Eq. 4.47a) solely on the conducting parts of the ~~inside surface of~~ shell (denoted as $\sum s_e$). The mentioned surfaces can be the electrodes of flow channel in a MHD device, or cross-sections of leads entering the interior of conventional electric machine. Everywhere on the remaining parts of the shell s , we have $\bar{I} = 0$.

The energy transfer by means of viscosity and diffusion can occur through the inlets and outlets of flow channels only. Since in the discussed systems those effects are negligible as compared with other kinds of transfer, we shall leave out the terms $(\bar{J}_d d^2\bar{s})$ and $(\bar{\Pi}_v \bar{u} d^2\bar{s})$ completely. On the other hand, the conductive-radiative transfer has to be fully considered, because all over the shell s ,

the flux-densities \bar{J}_{tc} and \bar{J}_{tr} do not usually vanish.

Taking all this into consideration and using substitutions (7.2a), (5.52b) and (4.47a), we list the particular kinds of transfer and the places of their occurrence:

$$\begin{aligned}
 (7.16a) \quad (\varrho \epsilon_s \bar{u} \, d^2\bar{s}) &= \begin{cases} \varrho \{ (\bar{u}^2/2) + \bar{I} + \epsilon_I \} (\bar{u} \, d^2\bar{s}) & \text{on } \sum s_f \\ 0 & \text{on remaining } s \end{cases} \\
 (7.16b) \quad (\bar{\Pi}' \bar{u} \, d^2\bar{s}) &= \begin{cases} (p \bar{u} \, d^2\bar{s}); \quad (\bar{\Pi}'_v \bar{u} \, d^2\bar{s}) = 0 & \text{on } \sum s_f \\ (\bar{\Pi}'_e \bar{u} \, d^2\bar{s}) & \text{on } \sum s_r \\ 0 & \text{on remaining } s \end{cases} \\
 (7.16c) \quad (\bar{J}_{ec} \, d^2\bar{s}) &= \begin{cases} (\psi \bar{I} \, d^2\bar{s}) & \text{on } \sum s_e \\ 0 & \text{on remaining } s \end{cases} \\
 (7.16d) \quad (\bar{J}_d \, d^2\bar{s}) &= 0 \quad \text{all over } s \\
 (7.16e) \quad (\{\bar{J}_{tr} + \bar{J}_{tc}\} \, d^2\bar{s}) &\neq 0 \quad \text{all over } s
 \end{aligned}$$

Considering the above equations and substituting Eq.(7.10b), we make the integral (7.15) more specific. We change its sign in order to present each of the terms as the net inflow of energy into the system (if the scalar obtained is negative, then it signifies the factual net outflow):

$$(7.17) \quad - \underbrace{\iint_{\sum s_r} (\bar{\Pi}'_e \bar{u} \, d^2\bar{s})}_{\text{I}} - \underbrace{\iint_{\sum s_f} (\varrho H' \bar{u} \, d^2\bar{s})}_{\text{II}} - \underbrace{\iint_{\sum s_e} (\psi \bar{I} \, d^2\bar{s})}_{\text{III}} - \underbrace{\oint_s (\{\bar{J}_{tc} + \bar{J}_{tr}\} \, d^2\bar{s})}_{\text{IV}} = 0$$

I	II	III	IV
the stress	convective	conductive	conductive-
net inflow	net inflow	net inflow	-radiative
of kinetic	of dynamic	of electro-	net inflow
energy	enthalpy	-magnetic	of internal
(delivered	(delivered)	energy	energy
mechanical		(delivered	(delivered
power)		electric	heat)
		power)	

for stationary process in electro-roto-flow system

According to Eq.(5.50c), the mechanical power (term I) equals the difference in torque on cross-sections of the outer ends of rotor times its angular velocity.

At approximately uniform velocity distribution over the surfaces

of inlet and outlet of the channel, the convective net inflow of dynamic enthalpy (term II) can be expressed as the product of the mass-flow rate and the drop $\Delta H'$ in the mass-density of dynamic enthalpy.

It is possible to show that, in case of the direct current, the electric power (term III) is equal to the product of the voltage drop $\Delta \varphi$ (difference in potential between the terminals) and intensity of the current flowing in the circuit.

In contrary to the first three terms of Eq.(7.17), which are assigned to particular surfaces, the computing of heat transfer (term IV) is rather complicated.

In a pure flow system (without rotor and electric circuit), only the terms II and IV are non-zero valued, while in the electro-flow system (without rotor) the term III comes in addition. In the MHD generator, the dynamic enthalpy is delivered to the system (term II > 0), whereas the electric power is taken away (term III < 0) and vice versa in the MHD pump.

In the electro-roto system (without channel), the terms I, III, and IV are non-zero valued. The generator (dynamo) is characterized by the positive term I and negative term III, while in the electric motor the signs are reversed. In the roto-flow system (without electric circuit), only the terms I, II and IV appear. For the turbine, the term II is positive and the term I negative, while for the rotary compressor the signs are reversed.

In all the rotor systems and electro-flow systems, the heat is usually taken away (term IV < 0).

7.6. The Balance of Entropy. The Second Law of Thermodynamics.

In the most general meaning, the entropy is a measure of the disorder of individual states. In the statistical thermodynamics, the entropy acquires a specific physical meaning as function of

probability of thermodynamic state.

In the phenomenological thermodynamics, the entropy is an axiomatic function of state, determining the way of occurrence of all the processes involving short-range interaction. Since such processes appear always in the substance, the entropy is regarded as a scalar SQ of some storage mass-density S , in thermodynamics called specific entropy. In a multicomponent mixture, the i -th entropy is regarded as a scalar CQ of the partial density S_i .

In contrary to such thermodynamic parameters as the volume, pressure, or temperature, which are close to our sensorial conceptions and "materially concrete", the entropy is of rather abstractive nature and seems to be a "spirit governing the matter". Regardless of our philosophical view, such a "spirit" does exist indeed in the Nature, because the short-range interactions are steered by definite tendencies.

Consider a fluid mixture being in the state of thermodynamic equilibrium. Basing on the Gibbs phase rule, which determines the number of independent parameters of state, we postulate a fundamental relation making the mass-density of internal energy \mathcal{E}_I dependent on the specific entropy S , specific volume v , and mass-fractions c_i of all n components of mixture: $\mathcal{E}_I\langle S, v, c_1 \dots i \dots n \rangle$.

The total differential of this fundamental relation is known as the Gibbs equation:

$$(7.18) \quad d\mathcal{E}_I = T dS - p dv + \sum \mu_i dc_i$$

where T , p , and μ_i have been thermodynamically defined as the following partial derivatives (the suffixes at square brackets symbolize the quantities kept constant while differentiation):

$$(7.19a) \left[\frac{\partial \epsilon_I}{\partial S} \right]_{v, c_i} = T, \quad (7.19b) - \left[\frac{\partial \epsilon_I}{\partial v} \right]_{S, c_i} = p$$

(absolute) temperature (absolute) pressure

$$(7.19c) \left[\frac{\partial \epsilon_I}{\partial c_i} \right]_{S, v, c_{j \neq i}} = \mu_i = H_i - TS_i \quad \begin{cases} \text{chemical potential} \\ \text{(partial density} \\ \text{of free enthalpy)} \\ \text{of } i\text{-th component} \end{cases}$$

The postulate I of nonequilibrium thermodynamics states that the definitions of all thermodynamic parameters, as well as the Gibbs equation, are valid in the state of local equilibrium, in other words, in the processes of sufficiently small time-space variability. Remembering this limitation, we can regard the increments in Eq.(7.18) as the substantial differentials. Dividing the Gibbs equation in its substantial form by dt/ϱ , we obtain following relation

$$(7.20) \quad \varrho \frac{d\epsilon_I}{dt} \rightarrow \bar{u} = \varrho T \frac{dS}{dt} \rightarrow \bar{u} - \varrho p \frac{dv}{dt} \rightarrow \bar{u} + \varrho \sum \mu_i \frac{dc_i}{dt} \rightarrow \bar{u}$$

the left-hand side of which is the same as in the balance of internal energy (5.58a). Confronting both equations (Ex. 7.6) results in the balance of entropy at substantial point:

$$(7.21) \quad \underbrace{\varrho \frac{dS}{dt} \rightarrow \bar{u}}_{\substack{\text{rate of change} \\ \text{of storage} \gtrless 0}} = \underbrace{\frac{\Psi}{T}}_{\text{production (creation)} > 0} - \underbrace{\text{div } \bar{I}}_{\text{transfer} \gtrless 0}$$

where \bar{I} is the non-convective transfer-flux density of entropy

$$(7.22a) \quad \bar{I} = \frac{\bar{J}_{tc} + \bar{J}_{tr}}{T} + \sum s_i \bar{J}_{di}$$

and Ψ/T is the production density of entropy, Ψ being called the dissipation function. The latter consists of 6 terms:

$$(7.22b) \quad \Psi =$$

$$= \underbrace{G_{pm}}_I + \underbrace{(\bar{I} \cdot \bar{E}_u)}_{II} - \underbrace{\sum (\bar{J}_{di} \overline{\text{grad}} \mu_i)}_{III} - \underbrace{(\bar{I} \overline{\text{grad}} T)}_{IV} - \underbrace{(\bar{I} \cdot \overline{\text{grad}} \bar{u})}_V - \underbrace{\varrho \sum s_i \mu_i}_{VI}$$

From Eq.(7.22a) we find that the flow of internal energy must be accompanied with the flow of entropy. It results from comparison of Eqs.(5.60), (6.62), and (6.63) that the first term of Eq.(7.22a) is equal to \bar{J}_q''/T , which is the second-law heat flux divided by temperature. The second term $\sum_i S_i \bar{J}_{di}$ represents the diffusive transfer of entropy (the componential entropies are carried with mixture's components in their diffusive motion).

Interpreting particular terms of the dissipation function (7.22b) we find that the production of entropy is connected with the following phenomena: electromagnetic hysteresis (I), flow of electric current (II), diffusion of mass (III, and partly IV, see Eq.7.22a), thermal conduction and radiation (IV), viscosity (V), and chemical, electric, or phase reaction (VI).

The use of the balance terminology allows to formulate the Second Law of Thermodynamics in the most concise and universal form, being the inequality

$$(7.23) \quad \bar{\Psi}/T > 0$$

According to this law, the entropy can only be created. It does not mean, however, that the storage of entropy must increase always and everywhere (compare the inequality signs below the balance Eq. 7.21). If only the outflow of entropy ($\text{div } \bar{I} > 0$) is larger than the creation, then the storage of entropy decreases ($\rho \frac{dS}{dt} \bar{u} < 0$).

It is easy to show (Ex. 7.7) that the storage of entropy must increase in the only case of thermodynamically isolated system (closed and adiabatic), because $\bar{I} = 0$ all over the shell $s_{\bar{u}}$ of such a system:

$$(7.24) \quad \frac{d}{dt} \bar{u} \iiint_{\substack{V \\ \rightarrow \bar{u}}} \rho S \, d^3V = \iiint_{\substack{V \\ \rightarrow \bar{u}}} \frac{\bar{\Psi}}{T} \, d^3V > 0 \quad \text{for isolated system}$$

Exercise 7.6. Derive the balance of entropy (7.21) and Eqs. (7.22a,b).

Instructions. We compare Eqs.(7.20) and (5.58a), divide the result by T and apply Eq.(6.16b), in order to obtain the intermediate equation

$$\varrho \frac{dS}{dt} \rightarrow \bar{u} = \frac{1}{T} \left\{ G_{pm} + (\bar{E}_{\rightarrow \bar{u}} \bar{I}_{\rightarrow \bar{u}}) - (\bar{N}_V \bar{g} \bar{r} \bar{a} \bar{d} \bar{u}) - \varrho \sum_i \bar{S}_i \mu_i \right\} + \\ - \frac{1}{T} \operatorname{div} \bar{J}_{nc} + \frac{1}{T} \sum_i \mu_i \operatorname{div} \bar{J}_{di}$$

We transform the last two terms according to rules (1.30a) and (1.28c). We apply substitutions (5.60), (6.62), and (7.19c), in order to obtain the expression (7.22a), and then Eqs.(7.22b) and (7.21).

Exercise 7.7. Prove the inequality (7.24).

Instructions. Basing on the equivalence of Eqs.(5.19a) and (5.18a), we transform the entropy balance at substantial point (7.21) into the balance in substantial region, then we substitute $\bar{I} = 0$ and apply the law (7.23).

7.7. Phenomenological Relations

It is worth to notice that the dissipation function (7.22b) is a sum of scalar products of two vectors (terms I, II, III, IV), two tensors (term V), and two scalars (term VI). The nonequilibrium thermodynamics makes an assumption that each term of the dissipation function is a product of particular thermodynamic flux by particular thermodynamic force. For example $\bar{I}_{\rightarrow \bar{u}}$, \bar{J}_{di} , \bar{I} , \bar{N}_V , $\varrho \bar{S}_i$ can be regarded as fluxes, and $\bar{E}_{\rightarrow \bar{u}}$, $-\bar{g} \bar{r} \bar{a} \bar{d} \mu_i$, $-\bar{g} \bar{r} \bar{a} \bar{d} T$, $-\bar{g} \bar{r} \bar{a} \bar{d} \bar{u}$, $-\mu_i$ as forces. The force and flux that appear in the same term of the dissipation function (e.g. $\bar{I}_{\rightarrow \bar{u}}$ and $\bar{E}_{\rightarrow \bar{u}}$) are conjugated.

The postulate II of nonequilibrium thermodynamics states that at not too intense time-space variability of fields, there exists a linear phenomenological relation between particular flux and all the forces (both conjugated and non-conjugated), provided the tensorial ranks of flux and force are identical or differ by 2 (the

Onsager relation and Curie's theorem).

If we disregard the cross phenomena occurring between non-conjugated fluxes and forces, then the postulate II leads to formulation of elementary phenomenological relations such as the simplified Ohm's law $\vec{E}_u = \kappa \vec{i}_u$ (Eq.4.45 at $\vec{K}_u = 0$), Fick's law of diffusion (6.13), and Newton's law of viscosity (5.26b).

When the cross phenomena cannot be neglected, some additional terms appear in the phenomenological relations. For example the current density \vec{i}_u depends also on the gradients of chemical potential and temperature, this effect being represented by vector \vec{K}_u in the full form of Ohm's law (4.45). The diffusion-flux density \vec{j}_{di} depends also on the electric field \vec{E}_u (electro-diffusion) and the temperature gradient (thermal diffusion, Soret's effect). The vector \vec{j}_{tc} of heat conduction can be influenced by electric field \vec{E}_u (Thomson's effect) and the chemical potential gradient (Dufour's effect).

The substitution of linear phenomenological relations makes the dissipation function (7.22b) a non-negative quadratic form, which satisfies the requirement of Second Law of Thermodynamics (7.23). Thus the processes governed by the phenomenological relations occur in such a way that the production of entropy is positive. We must, however, always bear in mind that the validity of phenomenological relations is restricted to relatively slow processes accompanied with small gradients of fields. The nonequilibrium thermodynamics deals with all those problems in detail.

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List of Letter Symbols

a (as upper affix)	antisymmetric part of a tensor
a	thermal diffusivity
\bar{a}	a tensor
$\overline{\text{adiv}}$ (as prefix)	antidivergence of a tensor
$\overline{\text{agr}}\overline{\text{ad}}$ (as prefix)	antigradient of a vector
$\overline{\text{arot}}$ (as prefix)	anticurl of a tensor
\bar{A}	(1) a vector, (2) electromagnetic vector potential
\bar{b}	a tensor
\bar{B}	(1) a vector, (2) magnetic induction
c	velocity of light in vacuum (as a constant)
\bar{c}	wave-velocity
c_i	i -th mass-fraction (concentration)
c_p, c_v	isobaric and isochoric specific heat, respectively
\bar{c}	a vector
CQ	componential quantity
d (as upper affix)	deviatorial part of a tensor
$\text{div}, \overline{\text{div}}$ (as prefix)	divergence of a vector and tensor, respectively
D	diffusivity
\bar{D}	electric displacement
\bar{E}	electric field vector
EQ	extensive quantity
\bar{F}	volume-density of body force
\bar{F}_{ed}	volume-density of electrodynamic force
\bar{F}_g	volume-density of gravitational force
\bar{F}_i	volume-density of i -th body force
g	scalar terrestrial acceleration

g	gravitational acceleration (gravitational field intensity)
$\overline{\text{grad}}, \overline{\overline{\text{grad}}}$ (as prefix)	gradient of a scalar and vector, respectively
G_{pm}	volume-density of polarization conversion (hysteretic annihilation) of electromagnetic energy
GGO	Gauss-Green-Ostrogradsky (theorem of)
h	elevation
$\overline{\text{helm}}$ (as prefix)	Helmholtzian of a vector
H	mass-density of (static) enthalpy (specific enthalpy)
\overline{H}	(1) transfer-flux density of a scalar EQ or SQ, (2) magnetic field vector
H'	mass-density of dynamic enthalpy
H_i	partial density of i -th (static) enthalpy
\overline{H}_i	transfer-flux density of a scalar i -th CQ
$H_{\underline{a}}$	directional-transmission density of a scalar EQ
i (as suffix)	referring to i -th component of mixture, i -th componential quantity or property
\overline{i}	(electric) current density
I	geometric moment of inertia of grain in the polar fluid
\overline{I}	non-convective transfer-flux density of entropy
\overline{J}	mass-flux density
\overline{J}_{di}	i -th (mass) diffusion-flux density
\overline{J}_i	i -th mass-flux density
\overline{J}	transfer-flux density of total energy
\overline{J}_d	diffusive transfer-flux density of internal energy

\bar{J}_{ec}	conductive transfer-flux density of electromagnetic energy
\bar{J}_{er}	radiative transfer-flux density of electromagnetic energy
\bar{J}_{nc}	non-convective transfer-flux density of internal energy
\bar{J}_{pc}	paraconvective transfer-flux density of total energy
\bar{J}'_{pc}	as above, but with electromagnetic terms excluded
\bar{J}'_q, \bar{J}''_q	first-law and second-law heat flux, respectively
\bar{J}_{tc}	conductive transfer-flux density of internal energy (thermal conduction)
\bar{J}_{tr}	radiative transfer-flux density of internal energy (thermal radiation)
\bar{K}	(electro-) motive vector
l	line, contour
$\lim f$ (as prefix)	apparent limit (limes fictus)
L	(macroscopic) characteristic linear dimension
m	mass
m_i	i-th mass
m_ν	individual mass of ν -th grain
n	total number of (1) grains in a set, (2) components in a mixture
p	pressure
p_i	i-th pressure
\bar{P}	storage volume-density of a vector EQ or SQ
\bar{P}_i	storage volume-density of a vector i-th CQ
\bar{P}, \bar{P}	production of a scalar and vector EQ, respectively
\bar{Q}	storage mass-density of a vector SQ
\bar{Q}_i	storage partial density of a vector i-th CQ

\bar{r}	position (-vector)
\bar{r}_ν	individual position of ν -th grain
reyn, $\overline{\text{reyn}}$ (as prefix)	Reynoldsian of a scalar and vector, respectively
$\overline{\text{rot}}, \overline{\text{rot}}$ (as prefix)	curl of a vector and tensor, respectively
\bar{R}	production volume-density of a vector EQ or SQ
\bar{R}_i	production volume-density of a vector i-th CQ
s (as upper affix)	symmetric part of a tensor
s	surface, shell
S	mass-density of entropy (specific entropy)
\bar{S}	production mass-density of a vector SQ
S_i	partial density of i-th entropy
\bar{S}_i	production partial density of a vector i-th CQ
SQ	substantial quantity
$\mathcal{S}, \bar{\mathcal{S}}$	storage of a scalar and vector EQ, respectively
t	time
$\overline{\text{thom}}$ (as prefix)	Thomsonian of a vector
T	temperature
\bar{T}	transfer-flux density of a vector EQ or SQ
\bar{T}_i	transfer-flux density of a vector i-th CQ
\bar{T}_n	directional-transmission density of a vector EQ
Tr (as prefix)	trace of a tensor
$\mathcal{T}, \bar{\mathcal{T}}$	transfer of a scalar and vector EQ, respectively
\bar{u}	barycentric velocity (of substance)
\bar{u}_i	i-th componential velocity
\bar{u}_ν	individual velocity of ν -th grain

\bar{u}_ν^*	individual thermal velocity of ν -th grain
v	specific volume (mass-density of volume)
$v_{\bullet i}$	partial specific volume (partial density of i -th volume)
V	spatial region, volume
$\overline{\text{Vec}}$ (as prefix)	(pseudo-) vector of a tensor
\bar{w}	referential velocity
\bar{w}_c	velocity of convective region
W	volume-density of total energy
W_{em}	volume-density of electromagnetic energy
x, y, z	Cartesian coordinates
x, y, z (as suffixes)	component referring to coordinate x, y, z , respectively
α	a scalar
β	(1) a scalar, (2) general diffusion coefficient
χ	a scalar
Γ	gravitational potential (mass-density of gravitational energy in terrestrial conditions)
δ (as prefix)	small interval or increment
$\delta l, \delta^3 m, \delta^3 m_i, \delta t, \delta^3 V$	intervals of phenomenological averaging (see symbols l, m, m_i, t, V)
$\delta \bar{l}$	small line segment (vector)
$\delta \bar{r}$	small deformative displacement (vector)
Δ (as prefix)	interval or increment
ε	storage mass-density of a scalar SQ
ε_d	mass-density of diffusion energy of mixture
ε_{di}	partial density of i -th diffusion energy
ε_e	mass-density of elastic energy
ε_i	storage partial density of a scalar i -th CQ
ε_I	mass-density of internal energy
ε_{Ii}	partial density of i -th internal energy

ϵ_{ki}	partial density of i-th kinetic energy
ϵ_s	mass-density of substantial energy
ρ_i	production mass-density of i-th mass
η_b, η_s	bulk and shear (dynamic) viscosity coefficients, respectively
η'_s	shear viscosity coefficient, according to traditional convention
η_1, η_2	first and second viscosity coefficients
Θ	(macroscopic) characteristic time of increase of a process
\mathcal{R}	(electric) resistivity
λ	(1) characteristic linear microdimension, (2) thermal conductivity
μ	(1) production mass-density of a scalar SQ, (2) (magnetic) permeability
μ_b, μ_s	bulk and shear Lamé's elasticity modulus, respectively
μ_i	(1) production partial density of a scalar i-th CQ, (2) i-th chemical potential (partial density of i-th free enthalpy)
ν (as suffix)	referring to ν -th individual grain
$\bar{\pi}$	stress (as non-convective transfer-flux density of momentum)
$\bar{\pi}'$	stress, in traditional convention
$\bar{\pi}_d$	diffusive stress of mixture
$\bar{\pi}_{di}$	i-th diffusive stress
$\bar{\pi}_e$	elastic stress
$\bar{\pi}'_e$	elastic stress, in traditional convention
$\bar{\pi}_i$	i-th stress
$\bar{\pi}_p$	pressure stress
$\bar{\pi}_v$	viscous stress
$\bar{\pi}'_v$	viscous stress, in traditional convention

$\bar{\bar{\pi}}_{vi}$	i-th viscous stress
ρ	(volume-) density of mass
ρ_e	(volume-) density of (electric) charge
ρ_i	(volume-) density of i-th mass
τ	characteristic microtime
φ	(1) storage volume-density of a scalar EQ or SQ, (2) electromagnetic scalar potential
φ_i	storage volume-density of a scalar i-th CQ
ψ	production volume-density of a scalar EQ or SQ
ψ_i	production volume-density of a scalar i-th CQ
Ψ	dissipation function
$\bar{\omega}$	(1) vorticity of flow, (2) macroscopically averaged angular velocity of grains in the polar fluid
$\bar{\omega}_o$	angular velocity of rigid region or body
Ω	solid angle
Ω (as suffix)	referring to direction of transport
$\bar{\mathbf{i}}_A$	unit vector of a vector $\bar{\mathbf{A}}$
$\bar{\mathbf{i}}$	unit tensor
$*$ (as suffix)	referring to actually instantaneous or local field
$*$ (as upper affix)	referring to micro-field or micro-variable component

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