

Conditions for Digraphs Representation of the Characteristic Polynomial

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Abstract. This paper presents additional conditions that are needed to create proper digraphs representation of the characteristic polynomial. Contrary to currently used methods (like canonical forms) digraphs representations allow to find a complete set of all possible realisations instead of only a few realisations. In addition, all realisations in the set are minimal. Proposed additional conditions on creating digraphs representations allow faster creation of representations by restricting the creations of representations that are not proper and would have to be removed in later steps of algorithm.

Keywords: Characteristic polynomial, realisation problem, dynamic system, digraphs.

Introduction

In recent years, linear positive systems are of great interest for many researchers. Analysis of the positive two-dimensional (2D) systems is more difficult than of positive one-dimensional (1D) systems, as additional problems arise in positive two-dimensional systems, that are not completely solved; for example: positive realisation problem [1], [2], [3], determination of lower and upper index reachability [4], [5], determination of reachability index set [6], [7], [8], etc.

The realisation problem is a very difficult task. In many research studies we can find canonical form of the system [2], [1], i.e. constant matrix form, which satisfies the system described by the transfer function. Use of that form allows us to write only one realisation of the system, while absolutely there exists many possible solutions. This means that there exist many sets of matrices which fit into system transfer function.

The digraphs theory was applied to the analysis of dynamical systems. The use of multidimensional theory was proposed for the first time in the paper [9] to analysis of positive two-dimensional systems. In [10] and [11] an experimental algorithm for finding set of possible realisations of the characteristic polynomial was proposed, but due to complicated nature of the problem (which is assumed to be NP-complete) it tends to find improper solutions, furthermore practical implementation is slow as the algorithm struggles with creating and eliminating many representations. In this article we propose lemma stating conditions under which digraphs representation is both proper and minimal for given characteristic polynomial, which allows to restrict the created representation set and speed-up the algorithm.

2D positive systems

Let $\mathbb{R}_+^{n \times m}$ be the set of $n \times m$ real matrices with non-negative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n \times 1}$. The set of non-negative integers will be denoted by \mathbb{Z}_+ and the $n \times n$ identity matrix by \mathbf{I}_n .

Consider the two-dimension (2D) general model described by the equation:

$$\begin{aligned} x_{i+1,j+1} &= \mathbf{A}_0 x_{ij} + \mathbf{A}_1 x_{i+1,j} + \mathbf{A}_2 x_{i,j+1} + \\ &\quad + \mathbf{B}_0 u_{ij} + \mathbf{B}_1 u_{i+1,j} + \mathbf{B}_2 u_{i,j+1} \\ y_{ij} &= \mathbf{C} x_{ij} + \mathbf{D} u_{ij} \end{aligned} \quad (1)$$

where $x_{ij} \in \mathbb{R}^n$, $u_{ij} \in \mathbb{R}^m$ and $y_{ij} \in \mathbb{R}^p$ are state, input and output vectors, respectively at the point (i, j) , and $\mathbf{A}_k \in \mathbb{R}^{n \times n}$, $\mathbf{B}_k \in \mathbb{R}^{n \times m}$, $k = 0, 1, 2$, $\mathbf{C} \in \mathbb{R}^{p \times n}$, $\mathbf{D} \in \mathbb{R}^{p \times m}$.

In this paper we will consider special case of general model for $\mathbf{A}_0 = 0$ and $\mathbf{B}_0 = 0$ – the second Fornasini-Marchesini model described by the equation:

$$\begin{aligned} x_{i+1,j+1} &= \mathbf{A}_1 x_{i+1,j} + \mathbf{A}_2 x_{i,j+1} + \\ &\quad + \mathbf{B}_1 u_{i+1,j} + \mathbf{B}_2 u_{i,j+1} \\ y_{ij} &= \mathbf{C} x_{ij} + \mathbf{D} u_{ij} \end{aligned} \quad (2)$$

For two-dimensional system the characteristic polynomial consist from two variables: z_1 and z_2 if we have discrete time system; s_1 and s_2 if we have continuous time system; z and s if we have hybrid system. For discrete time system described by the equation (2) we have the following characteristic polynomial:

$$\begin{aligned} d(z_1, z_2) &= \det[\mathbf{I} z_1 z_2 - \mathbf{A}_1 z - \mathbf{A}_2 z_2] = \\ &= z_1^n z_2^n - \sum_{i=0}^n \sum_{j=0}^n d_{ij} z_1^i z_2^j = \\ &= z_1^n z_2^n - d_{n-1,n} z_1^{n-1} z_2^n - d_{n,n-1} z_1^n z_2^{n-1} - \\ &\quad \cdots - d_{10} z_1 - d_{01} z_2 - d_{00} \end{aligned} \quad (3)$$

for

$$n \leq i + j \leq 2n - 1, \quad i, j = 0, 1, \dots, n \quad (4)$$

Digraphs

A directed graph [12], [13] (or just digraph) \mathcal{D} consists of a non-empty finite set $\mathbb{V}(\mathcal{D})$ of elements called vertices and a finite set $\mathbb{A}(\mathcal{D})$ of ordered pairs of distinct vertices called arcs. We call $\mathbb{V}(\mathcal{D})$ the vertex set and $\mathbb{A}(\mathcal{D})$ the arc set of \mathcal{D} . We will often write $\mathcal{D} = (\mathbb{V}, \mathbb{A})$ which means that \mathbb{V} and \mathbb{A} are the vertex set and arc set of \mathcal{D} , respectively. The order of \mathcal{D} is the number of vertices in \mathcal{D} . The size of \mathcal{D} is the number of arc in \mathcal{D} . For an arc (v_1, v_2) the first vertex v_1 is its tail and the second vertex v_2 is its head.

Example 1 The digraph \mathcal{D} in Figure 1 have order $\mathbb{V}(\mathcal{D}) = \{v_1, v_2, v_3\}$ equal to 3 and size $\mathbb{A}(\mathcal{D}) = \{(v_1, v_2), (v_2, v_3), (v_3, v_1), (v_3, v_2), (v_2, v_2)\}$ equal to 5.

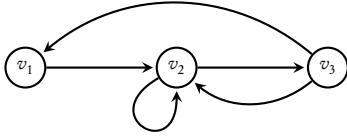


Fig. 1: A digraph \mathcal{D} .

A two dimensional digraphs $\mathcal{D}^{(2)}$ is a directed graph with two types of arcs (corresponding to matrices \mathbf{A} and \mathbf{A}_2) and input flows (corresponding to matrices \mathbf{B}_1 and \mathbf{B}_2). For the first time this type of digraphs was presented in paper [7].

Remark 1 \mathcal{A}_q -arcs and \mathcal{B}_q -arcs, are drawn by the other colour or line style. In this paper \mathcal{A}_1 -arc and \mathcal{B}_1 -arc is drawn by the solid line and \mathcal{A}_2 -arc and \mathcal{B}_2 -arc is drawn by the dashed line.

Conditions for digraphs realisation

Method proposed in [10] creates all possible digraphs representations for every monomial in the characteristic polynomial. After that one representation of each monomial is joined with others by means of disjoint union – algorithm repeats this step many times, to create all possible combinations of representations. Thus are created digraphs representations of the characteristic polynomial, which can be translated easily into A and B matrices. The problem with experimental algorithm is that, not all created representations are proper (it can be even said, that in many cases most representations are improper) and algorithm must eliminate them – but they can be checked only after the long process of creation. The idea is to find a set of restrictions, that will remove all improper representations, without the need of going through the time-costly representation creation process. Some assumptions were proposed earlier in [14] and [10], below we propose a set of restrictions tends that removes was proven experimentally to remove all improper results and retains all the proper representations.

Lemma 1 There exists positive state matrices of the positive system (2) corresponding to the characteristic polynomial (3) if

1. the coefficients of the characteristic polynomial are non-negative

$$d_{i,j} \geq 0, \quad \text{for } i, j = 0, 1, \dots, n; \quad d_{n,n} = 1 \quad (5)$$

2. digraph do not appear additional cycles.
3. the set A and B corresponding to two multidimensional digraphs are not disjoint.

To illustrate the workings of restrictions presented in the lemma let us consider the following example:

Example 2 Let the characteristic polynomial

$$d(z_1^{-1}, z_2^{-1}) = 1 - z_1^{-2} z_2^{-1} - z_1^{-1} z_2^{-1} - z_1^{-1} \quad (6)$$

be given and we need to determine its realisation.

In the first step we decompose polynomial (6) into a set of the simple monomials. We obtain:

- monomial $M_1 = z_1^{-1}$ (digraphs corresponding to monomial M_1 presented in Figure 2),
- monomial $M_2 = z_1^{-1} z_2^{-1}$ (digraphs corresponding to monomial M_2 presented in Figure 3),
- monomial $M_3 = z_1^{-2} z_2^{-1}$ (digraphs corresponding to monomial M_3 presented in Figure 4).



Fig. 2: Two dimensional digraphs corresponding to monomial M_1

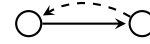


Fig. 3: Two dimensional digraphs corresponding to monomial M_2

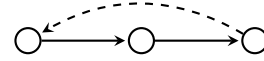


Fig. 4: Two dimensional digraphs corresponding to monomial M_3

In the next step we put together all simple monomial realisations presented in the Figure 2, Figure 3 and Figure 4. From all those realisations we choose the following three possible realisations, in which the monomial z_1^{-1} (colour red) presented in Figure 2 is toggled between vertices:

1. The first realisation is presented in Figure 5. From the digraphs obtained we can write state matrices A_1 and A_2 in the form of (7).

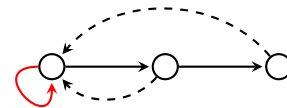


Fig. 5: The first two-dimensional digraphs

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} \textcolor{red}{1} & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (7)$$

Using Lemma 1 we check the conditions:

- (C₁) The coefficients of the characteristic polynomial (6) satisfy the condition (5). **The Condition 1 is satisfied.**
- (C₂) Obtained digraphs presented in Figure 5 do not create additional cycles. **The Condition 2 is satisfied.**
- (C₃) To verify this condition we must compare the sets A and B corresponding to representations of simple monomial digraphs.
- In the first step we compare set A (digraphs from Figure 4) corresponding to monomial $z_1^{-2}z_2^{-1}$ with the set B (digraphs from Figure 3) corresponding to monomial $z_1^{-1}z_2^{-1}$.

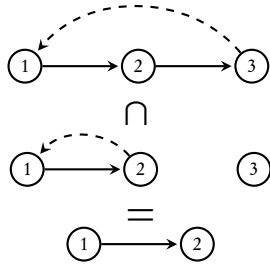


Fig. 6: $A \cap B$

- In the second step we compare set C (digraphs from Figure 6) with the set D (digraphs from Figure 2) corresponding to monomial z_1^{-1} .

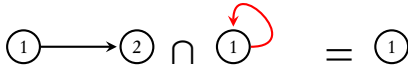


Fig. 7: $C \cap D$

As can be seen described realisation satisfy **Condition 3.**

All conditions are satisfied and digraphs presented in Figure 5 create one of possible realisations of the polynomial (6). Analytical verification of obtained solution is presented in equation (8).

$$\begin{aligned}
 d(z_1^{-1}, z_2^{-1}) &= \\
 &= \det \begin{bmatrix} 1 - z_1^{-1} & -z_2^{-1} & -z_2^{-1} \\ -z_1^{-1} & 1 & 0 \\ 0 & -z_1^{-1} & 1 \end{bmatrix} = \quad (8) \\
 &= 1 - z_1^{-1} - z_1^{-2}z_2^{-1} - z_1^{-1}z_2^{-1}
 \end{aligned}$$

2. The second realisation is presented in Figure 8. From obtained digraphs we can write state matrices A_1 and A_2 in the form of (9).

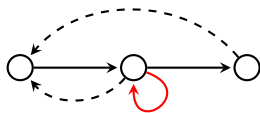


Fig. 8: The second two-dimensional digraphs

$$A_1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (9)$$

Using Lemma 1 we check the conditions:

- (C₁) The coefficients of the characteristic polynomial (6) satisfy the condition (5). **The Condition 1 is satisfied.**
- (C₂) Obtained digraphs presented in Figure 8 do not create additional cycles. **The Condition 2 is satisfied.**
- (C₃) To verify this condition we must compare the sets A and B corresponding to representations of simple monomial digraphs.
- In the first step we compare set A (digraphs from Figure 4) corresponding to monomial $z_1^{-2}z_2^{-1}$ with the set B (digraphs from Figure 3) corresponding to monomial $z_1^{-1}z_2^{-1}$. Solution is presented in Figure 6.
 - In the second step we compare set C (digraphs from Figure 6) with the set D (digraphs from Figure 2) corresponding to monomial z_1^{-1} .

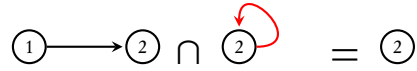


Fig. 9: $C \cap D$

Described realisation satisfies Condition 3.

All conditions are satisfied and digraphs presented in Figure 8 are one of possible realisations of the polynomial (6). Analytical verification of obtained solutions is presented in equation (10).

$$\begin{aligned}
 d(z_1^{-1}z_2^{-1}) &= \\
 &= \det \begin{bmatrix} 1 & -z_2^{-1} & -z_2^{-1} \\ -z_1^{-1} & 1 - z_1^{-1} & 0 \\ 0 & -z_1^{-1} & 1 \end{bmatrix} = \quad (10) \\
 &= 1 - z_1^{-1} - z_1^{-2}z_2^{-1} - z_1^{-1}z_2^{-1}
 \end{aligned}$$

3. The third realisation is presented in Figure 10. From obtain digraphs we can write state matrices A_1 and A_2 in the form of (11).

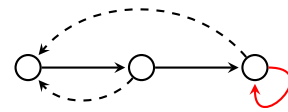


Fig. 10: The third two-dimensional digraphs

$$A_1 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad (11)$$

Using Lemma 1 we check the conditions:

- (C₁) The coefficients of the characteristic polynomial (6) satisfy the condition (5). **The Condition 1 is satisfied.**
- (C₂) Obtained digraphs presented in Figure 10 do not create additional cycles. **The Condition 2 is satisfied.**

(C₃) To verify this condition we must compare the sets A and B corresponding to representations of simple monomial digraphs.

- In the first step we compare set A (digraphs from Figure 4) corresponding to monomial $z_1^{-2}z_2^{-1}$ with the set B (digraphs from Figure 3) corresponding to monomial $z_1^{-1}z_2^{-1}$. Solution is presented in Figure 6.
- In the second step we compare set C (digraph from Figure 6) with the set D (digraph from Figure 2) corresponding to monomial z_1^{-1} .



Fig. 11: $C \cap D$

Described realisation does not satisfy Condition 3.

The third condition is not satisfied and digraphs presented in Figure 10 are improper realisations of the polynomial (6). Analytical verification of obtained solutions is presented in equation (12).

$$\begin{aligned}
 d(z_1^{-1}z_2^{-1}) &= \\
 &= \det \begin{bmatrix} 1 & -z_2^{-1} & -z_2^{-1} \\ -z_1^{-1} & 1 & 0 \\ 0 & -z_1^{-1} & 1 - z_1^{-1} \end{bmatrix} \\
 &= 1 - z_1^{-1} - z_1^{-2}z_2^{-1} - (1 - z_1^{-1})z_1^{-1}z_2^{-1} = \\
 &= 1 - z_1^{-1} - z_1^{-1}z_2^{-1}
 \end{aligned} \quad (12)$$

Concluding remarks

Proposed conditions for creating digraphs representations allow for faster creation of digraphs and matrices representations of the characteristic polynomial, as realisations can be checked during their creation. Application of proposed lemma to practical algorithm, proposed earlier in [10], would allow to restrict the set of created solutions only to those that are proper. The main advantages of proposed solutions in comparison to currently used methods is finding all possible realisations instead of just a few of them and the minimal form of A matrices. The lemma was proposed in the article was illustrated by numerical example, showing its workings.

ACKNOWLEDGMENT

Research has been financed with the funds of the Statutory Research of 2014.

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