

Characteristic Polynomial Realisation of Positive 2D Hybrid Linear Systems

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Abstract. In this paper, the new method of the determination of entries of the state matrices of the positive two-dimensional hybrid linear systems using multidimensional digraphs theory $\mathfrak{D}^{(n)}$ has been presented. For the proposed method parallel computing algorithm was constructed. Algorithm is based on GPGPU (General - Purpose Computing on Graphics Processing Units) computing method to gain needed speed and computational power for such solution. Proposed method discussed and illustrated by numerical examples. Proposed solution allows digraphs construction for any positive twodimensional system, regardless of their complexity.

Keywords: positive system, digraphs, algorithm, hybrid system.

Introduction

In recent years, many researchers have been interested in positive linear systems. Analysis of the positive two-dimensional (2D) systems is more difficult than of positive one-dimensional (1D) systems [1], [4], [5], [19], [12]. A lot of problems arise in positive two-dimensional systems, and they remain not completely solved; for example: positive realisation problem [15], [10], determination of index reachability [3], [2], [18], [7], [13], determination of reachability index set [8], [17], [11], etc.

In positive systems inputs, state variables and outputs take only non-negative values. Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated than standard systems. The realisation problem is very difficult task. In many research studies we can find canonical form of the system [15], i.e. constant matrix form, which satisfy the system described by the transfer function. With use of this form we are able to write only one realisation of the system. Absolutely, in general we have a lot of solutions. This means that we can find many sets of matrices which fit into system transfer function. The state of the art in positive systems theory is given in the monographs [4].

The digraphs theory was applied a little in the past to the analysis of dynamical systems. For the first time in the paper [6], [8] proposed the use of multidimensional digraphs theory to analysis of positive two-dimensional systems. Since then, more and more scientists try to use this theory in research. This work have been inspiration to use digraphs to solve realisation problem.

This work has been organized as follows: Chapter 2 present some notations and basic definitions of hybrid

systems and digraphs theory. In Chapter 3, we construct and discuss algorithm for determination of the set of polynomial realisations which based on digraphs theory and in Chapter 4 we illustrate it with numerical example. Finally we give some concluding remarks, present open problems and bibliography positions.

Preliminaries and problem formulation 2D Hybrid Systems

Let $\mathbb{R}_+^{n \times m}$ be the set of $n \times m$ matrices with nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n \times 1}$. The set of nonnegative integers will be denoted by \mathbb{Z}_+ and $n \times n$ identity matrix will be denoted by \mathbf{I}_n .

Consider a hybrid system described by the equations [14]:

$$\begin{aligned} \dot{x}_1(t, i) &= \mathbf{A}_{11}x_1(t, i) + \mathbf{A}_{12}x_2(t, i) + \mathbf{B}_1u(t, i) \\ (1) \quad x_2(t, i+1) &= \mathbf{A}_{21}x_1(t, i) + \mathbf{A}_{22}x_2(t, i) + \mathbf{B}_2u(t, i) \\ y(t, i) &= \mathbf{C}_1x_1(t, i) + \mathbf{C}_2x_2(t, i) + \mathbf{D}u(t, i) \\ t \in \mathbb{R}_+ &= [0, +\infty], \quad i \in \mathbb{Z}_+ \end{aligned}$$

where $\dot{x}_1(t, i) = (\partial x_1(t, i) / \partial t)$, $x_1(t, i) \in \mathbb{R}^{n_1}$, $x_2(t, i) \in \mathbb{R}^{n_2}$, $u(t, i) \in \mathbb{R}^m$, $y(i, j) \in \mathbb{R}^p$ and \mathbf{A}_{11} , \mathbf{A}_{12} , \mathbf{A}_{21} , \mathbf{A}_{22} , \mathbf{B}_1 , \mathbf{B}_2 , \mathbf{C}_1 , \mathbf{C}_2 , \mathbf{D} are real matrices.

Boundary conditions for system (1) have the form

$$x_1(0, i) = x_1(i), \quad i \in \mathbb{Z}_+, \quad x_2(t, 0) = x_2(t), \quad t \in \mathbb{R}_+$$

Definition 1 [14] *The hybrid system (1) is called internally positive if for all boundary conditions (2) and every sequence of inputs $u(t, i) \in \mathbb{R}^m$, $t \in \mathbb{R}_+$, $i \in \mathbb{Z}_+$ we have $x_1(t, i) \in \mathbb{R}^{n_1}$, $x_2(t, i) \in \mathbb{R}^{n_2}$, $t \in \mathbb{R}_+$, $i \in \mathbb{Z}_+$.*

Theorem 1 [14] *The hybrid system (1) is internally pos-*

itive if and only if

$$(2) \quad \begin{aligned} \mathbf{A}_1 &\in \mathbb{M}_{n_1}, \quad \mathbf{A}_{12} \in \mathbb{R}_+^{n_1 \times n_2}, \quad \mathbf{A}_{21} \in \mathbb{R}_+^{n_2 \times n_1}, \\ \mathbf{A}_{22} &\in \mathbb{R}_+^{n_2 \times n_2}, \quad \mathbf{B}_1 \in \mathbb{R}_+^{n_1 \times m}, \quad \mathbf{B}_2 \in \mathbb{R}_+^{n_2 \times m}, \\ \mathbf{C}_1 &\in \mathbb{R}_+^{p \times n_1}, \quad \mathbf{C}_2 \in \mathbb{R}_+^{p \times n_2}, \quad \mathbf{D} \in \mathbb{R}_+^{p \times m}. \end{aligned}$$

The transfer matrix of the system (1) is given by

$$(3) \quad \begin{aligned} \mathbf{T}(s, z) &= \\ &= \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \end{bmatrix} \begin{bmatrix} \mathbf{I}_{n_1}s - \mathbf{A}_{11} & -\mathbf{A}_{12} \\ -\mathbf{A}_{21} & \mathbf{I}_{n_2}z - \mathbf{A}_{22} \end{bmatrix}^{-1} \times \\ &\times \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} + \mathbf{D} \in \mathbb{R}^{p \times m}(s, z) \end{aligned}$$

In this paper we assume that the hybrid system describe by the equation (1) is SISO (Single-Input-Single-Output) system. In this case we can transfees matrix (3) rewritte in the following form

$$(4) \quad \begin{aligned} T(s, z) &= \frac{b_{n,m}s^m z^m + b_{n,m-1}s^n z^{m-1} + \dots}{s^n z^m - a_{n-1,m}s^n z^{m-1} - \dots} = \\ &= \frac{\dots + b_{11}sz + b_{10}s + b_{01}z + b_{00}}{\dots - a_{11}sz - a_{10}s - a_{01}z - a_{00}} \\ &= \frac{\sum_{i=0}^n \sum_{j=0}^m b_{i,j}s^i z^j}{s^n z^m - \left(\sum_{i=0}^n \sum_{j=0}^m a_{i,j}s^i z^j \right)} \end{aligned}$$

Multiplying numerator and denominator by of (4) by $s^{-n}z^{-m}$ we obtain:

$$(5) \quad \begin{aligned} T(s^{-1}, z^{-1}) &= \\ &= \frac{b_{n,m} + b_{n,m-1}z^{-1} + b_{n-1,m}s^{-1} + \dots}{1 - a_{n,m-1}z^{-1} - a_{n-1,m}s^{-1} - \dots} \\ &= \frac{\dots + b_{00}s^{-n}z^{-m}}{\dots - a_{00}s^{-n}z^{-m}} = \frac{N(s^{-1}, z^{-1})}{d(s^{-1}, z^{-1})} \end{aligned}$$

where

$$\begin{aligned} d(s^{-1}, z^{-1}) &= \\ &= 1 - a_{n,m-1}z^{-1} - a_{n-1,m}s^{-1} - \dots - a_{00}s^{-n}z^{-m} \end{aligned}$$

is the characteristic polynomial.

Digraphs

A multi-dimensional digraphs $\mathfrak{D}^{(n)}$ is a directed graph with n types of arcs and input flows. In detail, it is a $(\mathbb{S}, \mathbb{V}, \mathbb{X}_1, \mathbb{X}_2, \dots, \mathbb{X}_p, \mathbb{Y}_1, \mathbb{Y}_2, \dots, \mathbb{Y}_q)$, where $\mathbb{S} = \{s_1, s_2, \dots, s_m\}$ is the set of sources, $\mathbb{V} = \{v_1, v_2, \dots, v_n\}$ is the set of vertices, $\mathbb{X}_1, \mathbb{X}_2, \dots, \mathbb{X}_p$ are the subsets of $\mathbb{V} \times \mathbb{V}$ which elements are called \mathfrak{X}_1 -arcs and \mathfrak{X}_2 -arcs, \dots , \mathfrak{X}_p -arcs respectively, $\mathbb{B}_1, \mathbb{B}_2$ are the subsets of $\mathbb{S} \times \mathbb{V}$ which elements are called \mathfrak{Y}_1 -arcs and \mathfrak{Y}_2 -arcs, \dots , \mathfrak{Y}_q -arcs respectively where $p, q = 1 \dots \infty$.

The procedure for determmination multi-dimensional digraphs $\mathfrak{D}^{(n)}$ us the following:

- There exists \mathfrak{X}_1 -arc (\mathfrak{X}_2 -arc, \dots , \mathfrak{X}_p -arcs) from vertex v_j to vertex v_i if and only if the (i, j) -th entry of the matrix \mathbf{X}_1 ($\mathbf{X}_2, \dots, \mathbf{X}_p$) is nonzero.
- There exists \mathfrak{Y}_1 -arc (\mathfrak{Y}_2 -arc, \dots , \mathfrak{Y}_q) from source s_l to vertex v_j if and only if the l -th entry of the matrix \mathbf{Y}_1 ($\mathbf{Y}_2, \dots, \mathbf{Y}_q$) is nonzero.

Remark 1 \mathfrak{X}_1 -arc and \mathfrak{Y}_1 -arc are drawn by the other color than \mathfrak{X}_p -arc, and \mathfrak{Y}_q -arc where $p = q$. In this paper \mathfrak{X}_1 -arc, \mathfrak{Y}_1 -arc is drawn by the solid line and \mathfrak{X}_2 -arc and \mathfrak{Y}_2 -arc is drawn by the dashed line.

Example 1 The system described by the following matrices

$$(6) \quad (\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3) = \\ = \left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \right)$$

we can drew using multi-dimensional digraphs $\mathfrak{D}^{(n)}$ consisting of vertices v_1, v_2, v_3 and source s_1, s_2 . Multi-dimensional digraphs corresponding to system (6) is presented on Figure 1.

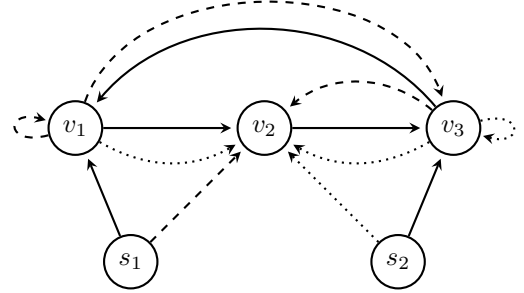


Fig. 1: Two-dimensional digraphs corresponding to system (6)

Problem Formulation

For the given positive hybrid systems system described by the model (1), we must determine all sets of realisations, which satisfy characteristic polynomial (6). The problem of finding all possible realisations of given polynomial is of such complexity, that it cannot be solved in reasonable time even by brute-force GPGPU method.

Problem solution

Proposed method finds all possible realisation of the characteristic polynomial (6) in two step. In the first step we decompose characteristic polynomial (6) on set of the simple monomials.

$$(7) \quad \begin{aligned} d(s^{-1}, z^{-1}) &= \\ &= 1 - d_{n,m-1}(s^{-1}, z^{-1}) - d_{n-1,m}(s^{-1}, z^{-1}) - \dots \\ &\dots - d_{00}(s^{-1}, z^{-1}) \end{aligned}$$

In the second step we can determine all possible characteristic polynomial realisation using all combinations of the digraph monomial representation determine in the first step. Parallel parts algorithms are realised with use of CUDA kernels. More about GPGPU computing method we can find in [16] and [9].

Numerical example

Let us consider the following example. For the given characteristic polynomial

$$(8) \quad d(s, z) = (s^2 + 3s - 1)(z^3 - 2z^2 - 4z - 3)$$

determine entries of the state matrices \mathbf{A}_{11} , \mathbf{A}_{12} , \mathbf{A}_{21} and \mathbf{A}_{22} using digraphs theory and GPGPU computing method. The above task we can divide on two subtask in the following form:

$$(9) \quad d(s) = s^2 + 3s - 1$$

$$(10) \quad d(z) = z^3 - 2z^2 - 4z - 3$$

Multiplying polynomial (9) by s^{-2} and polynomial (10) by z^{-3} we obtain

$$(11) \quad d(s^{-1}) = 1 + s^{-1} - s^{-2},$$

$$(12) \quad d(z^{-1}) = 1 - 2z^{-1} - 4z^{-2} - 3z^{-3}.$$

To solve this problem we use parallel algorithm.

Algorithm – createDigraphsKernel(V)

To determine all monomial realisation of the polynomial (12) in the first step we must determine all possible connections between vertices. In our example we have the following boundary conditions:

- number of vertices – $VN = 3$,
- number of colour in digraphs – $CN = 1$,
- monomial - $M_1 = [1]$ (corresponding to monomial z^{-1}), $M_2 = [2]$ (corresponding to z^{-2}), $M_3 = [3]$ (corresponding to z^{-3}).

For the monomial M_3 we have the following input

$$V = [0; \emptyset; [3]; 0; [1 \ 1 \ 1]]$$

Using the first part of the algorithm *createDigraphKernel()* we obtain the set of the possible connections between all vertices:

$$(13) \quad \begin{aligned} V[1,1] &= [0; [1 \ 0 \ 1]; [3]; 1; [0 \ 1 \ 1]]; \\ V[1,2] &= [0; [1 \ 0 \ 2]; [3]; 2; [1 \ 0 \ 1]]; \\ V[1,3] &= [0; [1 \ 0 \ 3]; [3]; 2; [1 \ 1 \ 0]]; \end{aligned}$$

Digraph $\mathfrak{D}^{(1)}$ corresponding to (13) presented on Figure 2. Using the second part of the algorithm

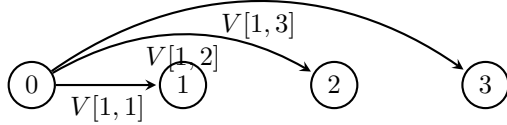


Fig. 2: One-dimensional digraphs corresponding to (13)

createDigraphKernel() we obtain the structure containing all the possible realisations of the monomial M_3 .

$$(14) \quad W = \left[1, \left[\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 1 \end{bmatrix} \right], \emptyset, 1, \emptyset \right] = V$$

Digraphs $\mathfrak{D}^{(1)}$ corresponding to (14) presented on Figure 3. In the same way we follow with monomial M_2 and M_3 and with polynomial (11).

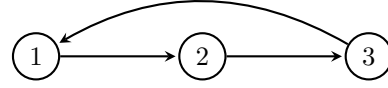


Fig. 3: One-dimensional digraphs corresponding to (14)

Algorithm - testSolutionKernel(R,cycles)

In the first step of the algorithm *creatingPolynomialRealisation(R,cycles)* we must write input structure: *cycles* and *arcs*:

$$(15) \quad cycles = [1 \ 1 \ 1],$$

$$arcs = \begin{bmatrix} [1 \ 0 \ 1] \\ [1 \ 1 \ 2] \\ [1 \ 2 \ 3] \\ [1 \ 3 \ 1] \\ [1 \ 1 \ 2] \\ [2 \ 2 \ 1] \\ [1 \ 1 \ 1] \end{bmatrix}.$$

Using the first part of the algorithm *creatingPolynomialRealisation()* we obtain the structure *arc_new* and matrix *P*:

$$(16) \quad \begin{aligned} arcs_new &= \begin{bmatrix} [1 \ 0 \ 1] \\ [1 \ 1 \ 2] \\ [1 \ 2 \ 3] \\ [1 \ 3 \ 1] \\ [1 \ 2 \ 2] \\ [1 \ 2 \ 1] \\ [1 \ 2 \ 1] \end{bmatrix}; \\ P &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \end{aligned}$$

Digraphs $\mathfrak{D}^{(1)}$ corresponding to matrix *P* described by equation (16) presented on Figure 2.

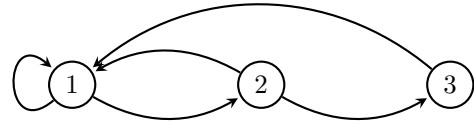


Fig. 4: One-dimensional digraphs corresponding to (16)

Using the second part of the algorithm *creatingPolynomialRealisation()* we check condition $\sum_{i=1, j=1, i=j}^{VN} P_{i,j} = cycles[1] = 1$. Condition is satisfied it means that we have simple cycle consisting of one vertex.

Using algorithm third part of the algorithm *creatingPolynomialRealisation()* we check for the cycles consisting of two vertices. In this step we create matrix *Q* by removing all rows and columns with the exception of *i*-th and *j*-th from matrix *P* and determine

product $p_{i,j} * p_{j,i}$.

$$(17) Q_{1,2} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = 1; Q_{1,3} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = 0;$$

$$Q_{2,3} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0; cycles[2] = 1 + 0 + 0 = 1$$

If condition is satisfied it means that we have simple cycle consisting of two vertices. Using algorithm forth part of the algorithm *creatingPolynomialRealisation()* we check for the cycles consisting of three vertices. In this step we create matrix Q and determine product $p_{i,j} * p_{j,i}$. We obtain $Q_3^{3 \times 3} = P = 1 = cycles[3]$.

At this moment we stop algorithm and we can say that digraphs presented on the Figure 4 satisfy polynomial (12). It should be noted that it is one of the possible realisations. To determine all polynomial realisations we should in the same way repeat algorithm for all combinations of the monomial realisations of M_1 , M_2 and M_3 . In this same way we determine realisation of the polynomial (11). Digraph corresponding to polynomial (11) presented on the Figure 5.



Fig. 5: Two-dimensional digraphs corresponding to polynomial (11)

Finally we write matrix \mathbf{A}_{11} and \mathbf{A}_{22} in the form:

$$(18) \mathbf{A}_{11} = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}; \mathbf{A}_{22} = \begin{bmatrix} 2 & 4 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Substituting obtained matrices: (18), $\mathbf{A}_{12} = \mathbf{A}_{21} = 0$ to (3) we obtain characteristic polynomial (8).

Concluding Remarks

The paper includes fast algorithm for determining all possible realisations of the characteristic polynomial of positive systems described with the use of the hybrid system which includes single input and single output (SISO). The proposed algorithm is based on the digraphs theory and GPGPU computing method. Currently, the method of determining a positive polynomial realisation using GPU units and digraphs methods is being implemented of the memory-efficient way. At the same time we are working on extension presented algorithm to solve reachability and realisation problems. Extending the proposed algorithm to dynamic systems of another class as well as searching for new areas of using multiprocessing calculations remains an open problem.

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