The Neumann problem in an irregular domain

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Abstract

We ask the question of patterns' stability for the reaction-diffusion equation with Neumann boundary conditions in an irregular domain in $\mathbb{R}^N$, $N \geq 2$, the model example being two convex regions connected by a small 'hole' in their boundaries. By patterns we mean solutions having an interface, i.e. a transition layer between two constants. It is well known that in 1D domains and in many 2D domains 'patterns' are unstable for this equation. We show that, unlike the 1D case, but as in 2D dumbbell domains, stable patterns exist. In a more general way, we prove invariance of stability properties for steady states when a sequence of domains Ω_n converges to our limit domain Ω in the sense of Mosco. We illustrate the theoretical results by numerical simulations of evolving and persisting interfaces.

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1   Introduction

Let us consider the Neumann boundary problem for the reaction–diffusion equation

$$u_t - \Delta u = g(u) \tag{1}$$

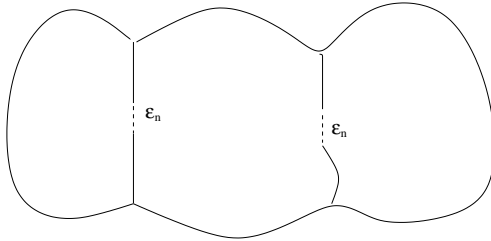on domains which have a crack, or 'splitting' inside: see Figure 1.

Figure 1: Our model domain Ω_n.

We are particularly interested in the steady, i.e. independent of time, solutions of (1), that we will also call equilibria or stationary states. Indeed, as part of the attractor, the steady states determine to a very large extent the evolution of any initial data. With respect to this process, the main feature of a steady solution is its stability, that is, roughly speaking, its being or not 'attractive' for other solutions in some neighborhood. This stability is expressed by the sign of the eigenvalues for the linearized problem.

It is well known that stability is strongly influenced by the domain's shape. And so, first of all, for Neumann boundary conditions and *convex* regions Ω, the only stable solutions are constants (given by the zeros of g). This has been proved in 1978 by Casten and Holland [CH] and independently in 1979 by Matano [M]. It is shown also in [CH] that this crucial property holds for a larger class of domains including annuli, and for all domains provided g is convex. On the other hand, Matano [M] constructs an example of connected region — of quite complex shape — for which there exist non-constant stable equilibria: they are nearly constant in some parts of the domain but possess also interfaces between these constant parts. In other words, he shows existence of a domain for which 'pattern formation' occurs with the reaction–diffusion equation.

Can this be expected for simpler shapes? Many works (Hale and Vegas [HV], Vegas [V], Jimbo [J1,J2], Jimbo and Morita [JM, MJ], de Oliveira et al. [OPP], Arrieta et al. [ACL] and references to previous works therein), addressed this question for dumbbell domains: two bigger regions connected by a thin strip. The answer, under various assumptions, is positive: stable interfaces exist. We want to ask here if the same occurs for 'split' regions, with cracks inside, as the one in Figure 1. We state below the conditions defining the class of our admissible domains.

Let us start with presenting some numerical experiments so as to illustrate the idea of this work. For the use of experiments, we have taken the example of the Allen-Cahn nonlinearity:

$$g(u) = u(1 - u^2),$$

so that (1) has two stable equilibria, 1 and -1, and one unstable, equal to 0. Figure 2 shows the results of three simulations. (We refer to the Appendix, Section 5, for

more details on the numerical method used here and for more investigations on the dynamics).
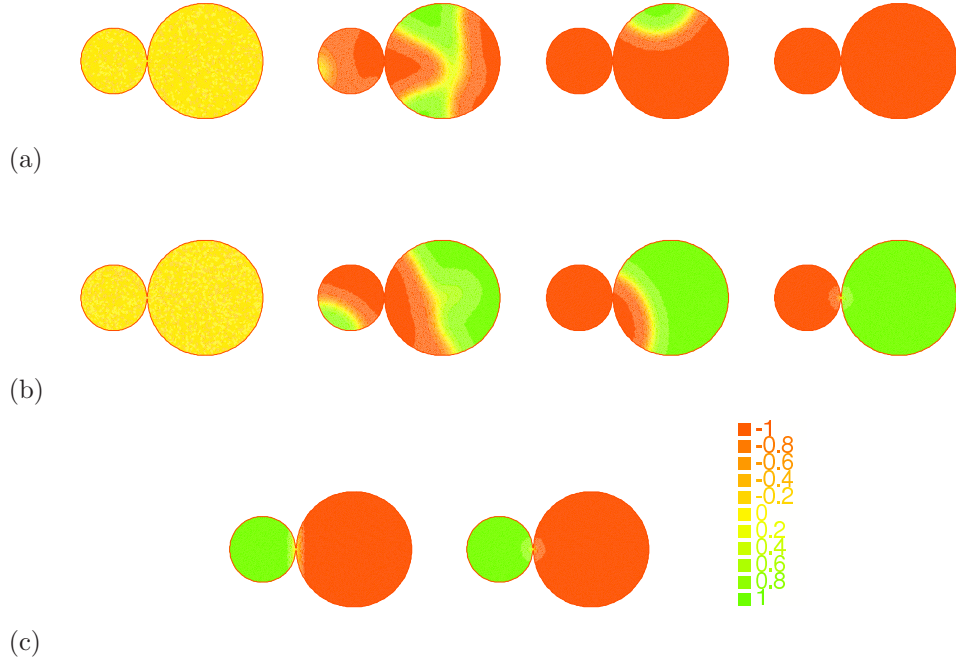

Figure 2: (a) the initial data $u_0 \approx 0$, with a random perturbation, on Ω_n. We visualize the evolution at $t = 0$, $t = 10$, $t = 40$ and $t = 100$. The last state is constant. (b) same experiment, the initial data $u_0 = 0$ being again randomly perturbed; $t = 0$, $t = 10$, $t = 100$ and $t = 200$. The last state has an interface at the level of the connection between the two subdomains. (c) $u_0 \approx 1$ on the left, $u_0 \approx -1$ on the right, $t = 0$ and $t = 1400$. The state possessing an interface seems not to evolve.

Experiments (a) and (b) are performed with initial data taken as a random perturbation of the constant 0, the stationary unstable solution. The system evolves towards two different states: a constant solution (equal to -1) in case (a) and a nonconstant one, close to ± 1 in each subregion in case (b). It is clear that the first one is stable. The experiment (c) is performed so as to verify the stability of the nonconstant steady state. We take there an initial datum equal to 1 in the domain on the left-hand side part of Ω_n and equal to -1 on the right-hand side part, except for a transition layer in a neighborhood of the connection, where it is linearly interpolated. This datum does not seem to evolve towards a constant state.

This visual impression is in no way a proof of any stability, not only because the domain is approximated, but even more in view of the known results about the extreme slowness of the evolution of the Allen-Cahn equation and of the 'dormant instability' of an analogous, nonconstant and monotone solution on a segment in one dimension, see Fusco and Hale [FH]. However, as we prove below, our two–dimensional geometry ensures stability of this solution.

We actually prove in this paper continuity of the stability properties of each
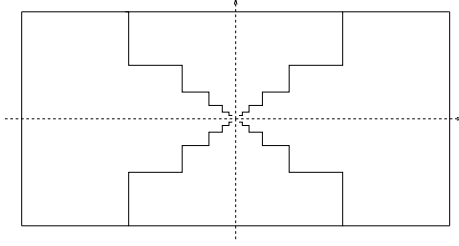
Figure 3: An admissible planar set Ω — the rectangle $(-2, 2) \times (-1, 1)$ deprived of four polylines: $\{(x, y) : 2^{-k} < |x| < 2^{1-k}, |y| = 2^{-k}$ or $|x| = 2^{-k}, 2^{-k-1} < |y| < 2^{-k}, k = 0, 1, 2\ldots\}$.

steady stable or unstable (hyperbolic) solution with respect to domain perturbations. In case of many usual nonlinearities g, having no degenerate zeros, we are able to count all steady solutions.

We will deal with the problem by considering it as a *perturbation* of a limit problem, posed on a set Ω which will typically be disconnected, but such that $\overline{\Omega}$ is connected. Also the domains Ω_n that we want to consider are not regular, their boundaries are not locally graphs of functions and they do not admit continuous extension operators $E : H^1(\omega) \to H^1(\mathbb{R}^n)$, $\omega = \Omega$ or Ω_n. In this point our work complements the paper of Arrieta and Carvalho [AC], which deals with the same problem for regular domains with lipschitz boundaries.

Let us list our main assumptions on Ω and its admissible perturbations and comment on the their concrete realizations.

(C_1) Ω is an open set, $(\Omega_n)_{n \in \mathbb{N}}$ a sequence of open sets and D a ball in $\mathbb{R}^n$ s. t.

$$\forall n \in \mathbb{N}, \quad \Omega \subset \Omega_{n+1} \subset \Omega_n \subset D;$$

(C_2) $\forall n \in \mathbb{N}, |\Omega_n| = |\Omega|$;

(C_3) the injection from $H^1(\Omega)$ into $L^2(\Omega)$ is compact;

(C_4) (Ω_n) converge in the sense of Mosco to Ω.

We also assume that g is a $C^1(\mathbb{R})$ function satisfying

(G) $\limsup\limits_{|x| \to +\infty} \dfrac{g(x)}{x} < 0.$

The geometrical sense of (C_1) and (C_2) is clear. (C_3) is a condition on $\partial\Omega$ regularity, but it is very weak — weak enough to allow splittings and disconnected Ω. Indeed, note first that if Ω has a finite number of connected components, compactness of the injection holds for Ω if it holds for its every connected component. On the other hand, as far as connected sets are concerned, it is satisfied by domains having the cone property, and, more generally, by domains being unions of a finite number of domains admitting an extension operator continuous in the H^1–norm. See the book of Maz'ya [Maz, Sections 1.6 and 1.10] for a review of results on this point and for examples. We give a nonstandard example of an admissible Ω in Figure 3, cf. [Maz, 1.5, p. 38].

Let us comment on (C_4). The definition of Mosco convergence is introduced rigorously in Section 2. Its essential feature is to be *equivalent* to convergence of

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solutions to the stationary Neumann problem on Ω_n to the solution of this problem on Ω, and this independently of the space dimension (see the paper of Dal Maso et al. [DMa] for the nonlinear setting and references therein for the linear one; the linear problem should be formulated in quotient spaces so as to have uniqueness). So, the use ofMosco convergence seems the most appropriate and general approach to our problem. However, as its definition is not geometrical, it is important to confront it with more intuitive notions of sets convergence. Indeed, for planar domains Ω_n having a bounded number of 'holes' (connected components of Ω_n^c) and converging to Ω in the Hausdorff complementary topology, it has been shown by Bucur and Varchon in [BV] that Mosco convergence is equivalent to the condition $\text{meas}(\Omega_n) \to \text{meas}(\Omega)$. This gives, directly, a very wide range of domains in $\mathbb{R}^2$ for which our results apply. In higher dimensions, Mosco convergence is more difficult to obtain, see e.g. Cortesani [Cor] who shows unstability of the linear Neumann problems satisfying the assumptions of [BV]; cf. also Damlamian [Dam] and many other works related to the Neumann sieve. However, it is known, see for instance Henrot [H1], that Mosco convergence occurs if the capacity of $\Omega_n \setminus \Omega$ converges to 0. (By capacity we mean here the 2–capacity; we refer to Evans and Gariepy [EG, Chapter 4] as for the notion and properties of capacity, which is a tool for measuring very fine sets. Let us just note here that all sets in $\mathbb{R}^N$ of Hausdorff dimension greater than $N-2$ are of non-zero capacity). Thus, one can see that (C_4) is satisfied for example if the connected components of Ω are each at zero distance from some other, and Ω_n is obtained by 'making holes' in the joining parts of the boundary, under the condition that the number of holes does not grow too rapidly and their size diminish. For most applications it is sufficient to assume that the number of holes remains constant.

The plan of this work is as follows. We begin with giving the main mathematical tools (Section 2), then state the main theorems on stability (Section 3) and finally we study convergence of the evolution problem (Section 4). In the main part, we show in Theorem 3.4, with the assumptions above, that that any hyperbolic, i.e. linearly stable or unstable steady state on Ω is a limit of a sequence $\{u_n\}$ of hyperbolic steady states on Ω_n, in the sense of $L^2(D)$ convergence. In Theorem 3.5, we show that, moreover, for n big enough, u_n has the same stability as u: the eigenvalues of the linearized operator $-\Delta - g'(u_n)$ converge to eigenvalues of the operator of $-\Delta - g'(u)$ (where Δ means the Neumann–laplacian and the second term is the multiplication operator). We also have convergence of all respective eigenspaces, still in in the sense of $L^2(D)$ distance (between unit eigenvectors).

Existence of non-constant stable equilibria on Ω_n, for big n, is an immediate consequence of Theorems 3.4 and 3.5. Indeed, take g with two stable zeros, like $g(u) = u(1-u^2)$ and put u to be equal to $+1$ and -1 on each connected component of Ω. This u forms a stable equilibrium which by our result has to be approached, in $L^2(D)$, by a sequence u_n of *stable* equilibria on Ω_n. What remains an open question, is the rate of this convergence: how small must the 'hole' be for the equilibria to become stable.

In the case when the system admits only hyperbolic equilibria, we show in Theorem 3.8 that their number is equal on Ω and on Ω_n. Theorem 3.9 states that in this case, the Hausdorff distance in $L^2(D)$ between the sets of stable steady points on Ω_n and the set of stable steady points on Ω is going to zero. The same is true for unstable equilibria.

Our method is based on the degree and operators' perturbation theories. The perturbation analysis is very close to the one performed in [AC], and the degree theory (the Leray–Schauder fixed point index) that we use in Theorem 3.4 has also been applied by the same authors in [ACL] for dumbbell domains. Also, the results that we obtain have their analogues in [AC], Propositions 3.1, 4.1 and Corollary 4.3. In [AC], continuity of unstable manifolds, and thus attractors, is also proved. In this point, however, the authors relay strongly on the existence of an extension operator continuous in H^1, which does not exist for our model domains. (Note that planar domains for which such an operator exists are known to be quasi–conformal to disks, see [Maz, Comments to Section 1.6] and references therein). Theorem 3.4 could also be deduced from the results of [DMa]; however, it seemed more simple to give a direct proof.

Finally, we devote Section 4 to the evolution problem. We estimate the difference of the semigroups in a norm containing an exponential weight with respect to time. This reflects the fact that, in spite of the results about steady states, the dynamics on Ω_n and on Ω are of course different.

2 Preliminaries

2.1 Main operator, linear problem

For all $f \in L^2(\Omega)$, the linear equation

$$\begin{cases} -\Delta u + u & = & f, & \text{in} & \Omega \\ \frac{\partial u}{\partial n} & = & 0 & \text{on} & \partial\Omega, \end{cases} \tag{2}$$

has a unique variational solution in $H^1(\Omega)$.

Definition 2.1. We will denote by A_Ω the Neumann–Laplacian operator $-\Delta + I$

$$A_\Omega \,:\, L^2(\Omega) \to L^2(\Omega)$$
$$A_\Omega v = -\Delta_N v + v$$

considered in the domain

$$D(A_\Omega) = \{u \in H^1(\Omega) : \ \exists f \in L^2(\Omega), \text{such that} \ u \ \text{solution of (2)}\}.$$

The operator A_Ω is selfadjoint, closed, has compact resolvent and its first eigenvalue equal to 1. A simple calculation using the canonical expansion in the basis of eigenvectors (cf [H2, 1.3]) shows that A_Ω satisfies

$$\left\| (\lambda - A_\Omega)^{-1} \right\|_{\mathcal{L}(L^2(\Omega))} \le \frac{2}{|\lambda - 1|}, \tag{3}$$

for all λ in the sector

$$\mathcal{S} = \{\lambda, \frac{\pi}{3} \le |arg(\lambda - 1)| \le \pi, \lambda \ne 1\},$$

i.e. it is sectorial in the sense of Henry [H2, Definition 1.3.1]. We have denoted by $\mathcal{L}(L^2(\Omega))$ the space of linear continuous operators in $L^2(\Omega)$. The operator $-A_\Omega$ is then (see [H2, Theorem 1.3.4]) the infinitesimal generator of an analytic linear semigroup $\{S_\Omega(t)\}_{t \ge 0}$, which can be written as

$$S_\Omega(t) = \frac{1}{2i\pi} \int_\Gamma e^{\lambda t} (\lambda + A_\Omega)^{-1} d\lambda, \tag{4}$$

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and Γ is a contour in the resolvent set $\rho(-A_\Omega)$, with $\arg\lambda \to \pm\theta$ as $|\lambda| \to \infty$ for some $\theta \in (\frac{\pi}{2}, \pi)$. We also have (cf [H2, Theorem 1.3.4])

$$\|S_\Omega(t)\|_{\mathcal{L}(L^2(\Omega))} \leq e^{-t}, \qquad \|S_\Omega(t)\|_{\mathcal{L}(L^2(\Omega), H^1(\Omega))} \leq \frac{e^{-t}}{\sqrt{t}}. \tag{5}$$

2.2 Definition and basic properties of solutions

For the needs of our analysis of the problem, we will now write (1) as

$$\begin{cases} u_t + A_\Omega u &= f(u), \quad t > 0 \\ u(0) &= u_0 \end{cases} \tag{6}$$

with

$$f = g + Id.$$

Note that the assumption (G) takes now the form

$$\limsup_{|x| \to +\infty} \frac{f(x)}{x} < 1, \tag{7}$$

Definition 2.2. By a solution of (1) on an open set Ω, with an initial condition $u_0 \in L^2(\Omega)$, we understand a continuous function from $[0, +\infty)$ into $L^2(\Omega)$, satisfying on $(0, +\infty)$ the following integral equation

$$u(t) = S_\Omega(t)u_0 + \int_0^t S_\Omega(t-s)f(u(s))ds. \tag{8}$$

with S_Ω given by (4). By a stationary solution (steady point, equilibrium) of (1)we mean a solution of:

$$A_\Omega u = f(u). \tag{9}$$

The set of stationary points will be denoted by $SP(\Omega)$.

Our notion of solution corresponds to what is often called 'mild' solution. Every solution in the sense of Definition 2.2 satisfies (6); that is, mild solutions are also variational (weak) solutions. This is shown by calculating $u(t+h) - u(t)$ and passing to the limit $h \to 0$ (cf [H2, Lemma 3.3.2]). Let us now consider the stationary states.

Definition 2.3. The linearization of $A_\Omega - f$ at $u \in SP(\Omega)$ is the closed, linear operator

$$A_\Omega - f'(u) : L^2(\Omega) \to L^2(\Omega)$$
$$\left(A_\Omega - f'(u)\right)v = A_\Omega v - f'(u)v$$
$$D(A_\Omega - f'(u)) = D(A_\Omega).$$

For all $u \in SP(\Omega)$ and $k \in \mathbb{N}$ let $\lambda_k(A_\Omega - f'(u))$ be the kth eigenvalue of the operator $A_\Omega - f'(u)$. Let:

$$SP^+(\Omega) = \{u \in SP(\Omega), \lambda_1(A_\Omega - f'(u)) > 0\}$$

be the subset of stable steady points,

$$SP^0(\Omega) = \{u \in SP(\Omega), \ \exists k \in \mathbb{N} \ : \ \lambda_k(A_\Omega - f'(u)) = 0\}$$

be the subset of non-hyperbolic steady points,

$$SP^-(\Omega) = \{u \in SP(\Omega), \ \lambda_1(A_\Omega - f'(u)) < 0, \ \forall k \in I\!N \ : \ \lambda_k(A_\Omega - f'(u)) \neq 0\}$$

be the subset of unstable steady points. The set $SP^+(\Omega) \cup SP^-(\Omega)$ is the subset of hyperbolic steady points. It is known (see [H2, Chapter 4]) that each of them is isolated in $L^2(\Omega)$. It is known also that stable steady states are attractive in the sense of Liapunov, and the unstable ones are repulsive for almost all data in their neighborhood.

The condition (7) implies that equilibria and that solutions of the parabolic equation with initial conditions in $L^\infty(\Omega)$ are uniformly bounded in $\mathrm{L}^\infty(\Omega)$.

Proposition 2.4. *There exists $K > 0$ such that*

$$\begin{aligned}
\forall u \in SP(\Omega) \qquad & \|u\|_{L^\infty(\Omega)} \leq K, \\
\forall t > 0, \qquad & \|u(t)\|_{L^\infty(\Omega)} \leq \max\{K, \|u_0\|_{L^\infty(\Omega)}\}
\end{aligned}$$

where $u(t)$ is the solution of (6) with the initial condition u_0.

Proof. The proof is standard and applies to all weak solutions, see e.g. [GT]. By (7), there exists $K > 0$ such that $\forall x : | : |x| > K, \ f(x)/x < 1$. Taking as test function $(u - k)^+$ and $(u + k)^-$ in (9), we obtain $-K \leq u \leq K$. The second statement is proved in the same way but using (1) and with $K' = \max(K, \|u_0\|_\infty)$. We conclude by the Gronwall lemma. $\qquad\square$

Let K be the constant given in Proposition 2.4, we denote by C_f^K the smallest constant such that

$$|f(x) - f(y)| \leq C_f^K |x - y| \ \ \forall x, y \in \overline{B}(0, K). \tag{10}$$

Of course f is also bounded on $B(0, K)$.

Corollary 2.5. *Under the conditions $f \in C^1(I\!R)$, (7) and $u_0 \in L^\infty(\Omega)$, we may assume that f is* globally *lipschitz continuous (and bounded). So, for all $u_0 \in L^\infty(\Omega)$, there exists a unique solution of (6).*

See e.g. [H2, Corollary 3.3.5]. This will be crucial in most proofs which follow.

2.3   Restriction and extension between D and $\Omega \subset D$

As we will perturb the domain, we need one large space in which solutions can be considered and compared: this is $L^2(D)$, D (the design region) being given by (C_1). Let us denote by $\|.\|$ the norm in $\mathcal{L}(L^2(D))$ and fix the following operators of restriction and extension:

$$r_\Omega \in \mathcal{L}(L^2(D), L^2(\Omega)) \quad \text{and} \quad p_\Omega \in \mathcal{L}(L^2(\Omega)), L^2(D))$$

defined by

$$\begin{aligned}
\forall u \in L^2(D), \quad & r_\Omega(u) = u \quad \text{in} \quad \Omega, \\
\forall u \in L^2(\Omega), \quad & p_\Omega(u) = u \quad \text{in} \quad \Omega \quad p_\Omega(u) = 0 \quad \in \Omega^c.
\end{aligned} \tag{11}$$

When applied to a vector, the operator p_Ω acts on each of its components.

Of course p_Ω is not continuous in $H^1(D)$. We note now that the operators $p_\Omega \circ (\lambda - A_\Omega)^{-1} \circ r_\Omega$ and $p_\Omega \circ S_\Omega(t) \circ r_\Omega$ belong to $\mathcal{L}(L^2(D))$ and the values of the

norms remain unchanged. So, we will consider the resolvent operator $(\lambda - A_\Omega)^{-1}$ as an operator from $L^2(D)$ to $L^2(D)$. When it does not lead into confusion, we will often, in what follows, omit the operators p_Ω and r_Ω. It is clear that the formula (4) remains valid when we consider it in the sense of composition with p_Ω and r_Ω.

2.4 Mosco convergence

We introduce now the notion of Mosco convergence which is our main assumption (C_4). As in other works related to the Neumann perturbation problem [DMa, Dan4], we use here actually a special case of the Mosco convergence as introduced in his original paper [Mo, Definition 1.1]. By applying this definition to linear subspaces of $L^2(D)^{N+1}$:

$$X_{\Omega_n} = \{(p_{\Omega_n}(u), p_{\Omega_n}(\nabla u)) \ : \ u \in L^2(\Omega_n)\}$$
$$X_\Omega = \{(p_\Omega(u), p_\Omega(\nabla u)) \ : \ u \in L^2(\Omega)\}.$$

we obtain the following

Definition 2.6. Let Ω be an open set and $(\Omega_n)_{n\in \mathbb{N}}$ a sequence of open sets. We say that $(\Omega_n)_{n\in \mathbb{N}}$ *converges in the sense of Mosco* to Ω if the following conditions (M_1) and (M_2) occur:

(M_1) if $u_n \in H^1(\Omega_n)$ are such that

$$p_{\Omega_n}(u_n) \xrightarrow{L^2(D)} v, \quad p_{\Omega_n}(\nabla u_n) \xrightarrow{[L^2(D)]^N} b,$$

then there exists $u \in H^1(\Omega)$ such that $v = p_\Omega(u)$ and $b = p_\Omega(\nabla u)$.

(M_2) for all $u \in H^1(\Omega)$, there exists $u_n \in H^1(\Omega_n)$ such that

$$p_{\Omega_n}(u_n) \xrightarrow{L^2(D)} p_\Omega(u), \quad p_{\Omega_n}(\nabla u_n) \xrightarrow{[L^2(D)]^N} p_\Omega(\nabla u).$$

As noted in the Introduction, it is known that under conditions (M_1) and (M_2), the solution to the linear problem (2) is continuous with respect to domain perturbations; the same was shown for semi–linear case in [DMa]. In Section 4, we will prove this also for solutions of the nonlinear evolution equation (6). We have given in the Introduction more known results about this convergence.

2.5 Distance between linear spaces

In order to stata some of the convergence results, we want to make precise the notion of distance between sets and between linear spaces.

Definition 2.7. (See [K, IV.2]). Let d^H be the symmetric Hausdorff distance between sets: if X and Y are nonempty subsets of a normed space $(Z, \|\cdot\|)$,

$$d^H(X, Y) = \max\Big(\sup_{x\in X} \inf_{y\in Y} \|x - y\|, \ \sup_{y\in Y} \inf_{x\in X} \|x - y\|\Big).$$

If X and Y are linear spaces, $d^H(X, Y)$ is infinite or 0. So, let d be the Hausdorff distance defined on the set of all unit vectors:

$$d(X, Y) = \max\Big(\sup_{\substack{x\in X \\ \|x\|=1}} \inf_{\substack{y\in Y \\ \|y\|=1}} \|x - y\|, \ \sup_{\substack{y\in Y \\ \|y\|=1}} \inf_{\substack{x\in X \\ \|x\|=1}} \|x - y\|\Big).$$

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If we complete this definition by setting:

$$d(X, \{0\}) = 2 \text{ for } X \neq \{0\}, \quad d(\{0\}, \{0\}) = 0,$$

then d is a distance on the set of all closed subspaces of Z.

Remark 2.8. 1) The distance d induces on the set of closed subspaces of Z the same topology as the 'gap' defined as

$$\delta(X, Y) = \max \Big(\sup_{\substack{x \in X \\ \|x\|=1}} \inf_{y \in Y} \|x - y\|, \ \sup_{\substack{y \in Y \\ \|y\|=1}} \inf_{x \in X} \|x - y\| \Big).$$

See [K, IV.2.1]. This 'gap' appears in the statements of many theorems in [K].
2) Note that $d(X, Y) < 1$ implies dim $X =$ dim Y [K, IV.2.2].

3   Stability

In all what follows in this part, we consider a domain Ω and a sequence of domains $(\Omega_n)_{n \in I\!N}$ satisfying conditions (C_1, C_2, C_3, C_4). For simplicity, let us note A_n, S_n, A, S instead of A_{Ω_n}, S_{Ω_n}, A_Ω, S_Ω and p_n, p instead of p_{Ω_n}, p_Ω.

3.1   Resolvents convergence

We denote by R_A the resolvent operator which to v associates $A^{-1}(f(v))$, extended by zero outside Ω:

$$R_A : L^2(D) \to L^2(D)$$
$$v \mapsto p\big(A^{-1}(f(v))\big). \tag{12}$$

We define R_{A_n} in the same way. Note that only $v|_\Omega$ enter into the definition of $R_A(v)$, and so we can consider $L^2(\Omega)$ as the effective domain of R_A. We also have

$$u \in SP(\Omega) \Leftrightarrow R_A(u) = u.$$

It is clear that R_A is a compact operator, verifying for all $v \in L^2(D)$

$$\|R_A(v)\|_{H^1(\Omega)} \leq C_f^K \|v\|_{L^2(D)}$$

where C_f^K is defined by (10). The following result will be essential for the proof of Theorem 3.4.

Lemma 3.1. *Suppose that (w_n) is a sequence in $L^2(D)$ such that $(f(w_n))$ converges weakly to h in $L^2(D)$. Then*

$$\lim_{n \to \infty} \|R_{A_n}(w_n) - A^{-1}(h)\|_{L^2(D)} = 0,$$
$$\lim_{n \to \infty} \|R_A(w_n) - A^{-1}(h)\|_{L^2(D)} = 0.$$

Proof of Lemma 3.1. Let us prove the first claim. Let $u_n = R_{A_n}(w_n)$:

$$\int_{\Omega_n} \nabla u_n \nabla \phi_n + u_n \phi_n = \int_{\Omega_n} f(w_n) \phi_n \tag{13}$$

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for all $\phi_n \in H^1(\Omega_n)$. It is easy to see that $(p_n(u_n), p_n(\nabla u_n))$ is bounded in $L^2(D)^{N+1}$. So, by (M_1),

$$(p_n(u_n), p_n(\nabla u_n)) \xrightarrow{[L^2(D)]^{N+1}} (p(u), p(\nabla u)).$$

Let (φ_n) and φ be given by (M_2) and take them as test functions in (13). We can now pass to the limit, obtaining $Au = h$ on Ω. So, (u_n) converges to $A^{-1}(h)$ weakly in $L^2(D)$. By (C_3), we obtain that (u_n) converges to $A^{-1}(h)$ strongly in $L^2(\Omega)$. With (C_2), this gives

$$\int_{\Omega_n} |u_n - A^{-1}(h)|^2 = \int_\Omega |u_n - A^{-1}(h)|^2 \to 0 \text{ as } n \to \infty. \tag{14}$$

This ends the proof of the first statement. The second one comes from the weak continuity of the operator A^{-1} and the compact injection of $H^1(\Omega)$ into $L^2(D)$, (C_3). $\qquad\square$

Remark 3.2. Lemma 3.1 remains true for domains which do not satisfy (C_2), but a more general condition $|\Omega_n \setminus \Omega| \to 0$. One should just replace (14) by

$$\lim_{n \to \infty} \int_{D \setminus \Omega} |u_n - A^{-1}(h)|^2 = 0.$$

This follows from boundedness of f, which gives uniform boundedness of $(u_n) = R_{A_n}(w_n)$ in $L^\infty(D)$.

Lemma 3.3. Let (h_n) be a sequence in $L^2(D)$ such that $\|h_n\|_{L^2(D)} \leq 1$. Let $\lambda > C_f^K - 1$. (We recall that f can be considered as being lipschitz continuous with the Lipschitz constant C_f^K). Then

$$\lim_{n \to \infty} \|(A_n + \lambda - f'(u_n))^{-1}(h_n) - (A + \lambda - f'(u))^{-1}(h_n)\|_{L^2(D)} = 0.$$

Proof of Lemma 3.3. Up to a subsequence, h_n converges weakly in $L^2(D)$ to h. Let

$$v_n = ((A_n + \lambda - f'(u_n))^{-1})(h_n),$$

then

$$\int_{\Omega_n} |\nabla v_n|^2 + \int_{\Omega_n} v_n^2 + \lambda \int_{\Omega_n} v_n^2 + \int_{\Omega_n} f'(u_n) v_n^2 = \int_{\Omega_n} h_n v_n,$$

which gives

$$\|v_n\|_{H^1(\Omega_n)}^2 \leq 1 + \frac{1}{1 + \lambda - C_f^K}.$$

So, by (M_1) there exists $v \in H^1(\Omega)$ such that, up to a subsequence,

$$(p_n(\nabla v_n), p_n(v_n)) \xrightarrow{[L^2(D)]^{N+1}} (p(\nabla v), p(v)).$$

Also, by (C_3), v_n converge strongly to v in $L^2(\Omega)$. Let $\varphi \in H^1(\Omega)$ and let $\varphi_n \in H^1(\Omega_n)$ be given by (M_2). Take them as test functions in the equation defining v_n:

$$\int_{\Omega_n} \{\nabla v_n \nabla \varphi_n + (1 + \lambda) v_n \varphi_n - f'(u_n) v_n \varphi_n\} = \int_{\Omega_n} h_n \varphi_n,$$

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for all $n \in \mathbb{N}$. Passing to the limit, with boundedness of f', we obtain that $v = (A + \lambda - f'(u))^{-1}(h)$. We use now v_n as test function in the above equation. Note that, by (C_2) and (C_3),

$$\int_{\Omega_n} h_n v_n = \int_\Omega h_n v_n \longrightarrow \int_\Omega hv.$$

Thus

$$\lim_{n\to\infty} \int_{\Omega_n} \{|\nabla v_n|^2 + (1+\lambda)|v_n|^2 - f'(u_n)|v_n|^2\} = \int_\Omega \{|\nabla v|^2 + (1+\lambda)|v|^2 - f'(u)|v|^2\}.$$

It follows that

$$\lim_{n\to\infty} \int_D \{|p_n(\nabla v_n) - p(\nabla v)|^2 + |p_n(v_n) - p(v)|^2\} = 0.$$

So,

$$\lim_{n\to\infty} \int_D |p_n(v_n) - p(v)|^2 = 0.$$

This means that $(A_n + \lambda - f'(u_n))^{-1}(h_n)$ converges to $(A + \lambda - f'(u))^{-1}(h)$ in $L^2(D)$. On the other hand, it is easy to see that $(A + \lambda - f'(u))^{-1}(h_n)$ converges to $(A + \lambda - f'(u))^{-1}(h)$ in $L^2(D)$. This ends the proof. $\qquad\square$

3.2 Main theorems

We state now continuity of the hyperbolic equilibrium point with respect to our domain perturbation.

Theorem 3.4. *For all $u \in SP^+(\Omega) \cup SP^-(\Omega)$, there exists $(u_n)_{n\in\mathbb{N}}$ which converges to u in $L^2(D)$ and such that $u_n \in SP(\Omega_n)$ for all n.*

Theorem 3.5. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence such that $u_n \in SP(\Omega_n)$ and which converges to u in $L^2(D)$. Then $u \in SP(\Omega)$ and for every $k \in \mathbb{N}$,*

$$\begin{aligned} \lambda_k(A_{\Omega_n} - f'(u_n)) &\longrightarrow \lambda_k(A_\Omega - f'(u)) &\text{as } n \to \infty, \\ d(W_n^k; W^k) &\longrightarrow 0 &\text{as } n \to \infty. \end{aligned}$$

Here d refers to the distance between closed subspaces of $L^2(D)$, as in Definition 2.7 and W^k, W_n^k are the subspaces generated by k first eigenvectors:

$$W^k = span[e^1; \ldots; e^k], \qquad W_n^k = span[e_n^1; \ldots; e_n^k]$$

where e^k is the eigenvector corresponding to the eigenvalue $\lambda_k(A_\Omega - f'(u))$, e_n^k the eigenvector corresponding to the eigenvalue $\lambda_k(A_{\Omega_n} - f'(u_n))$.

Without loss of generality, with Proposition 2.4 and Corollary 2.5, we can suppose that f is lipschitz continuous with $\mathrm{Lip}\, f = C_f^K$. Here again, we consider that the functions are extended by zero outside the open set in which they are naturally defined. To prove this result, we will use the Leray-Schauder Fixed-Point Index (see e.g. [Z, vol. I, chapter 12]).

Proof of Theorem 3.4. Let $u \in SP^+(\Omega) \cup SP^-(\Omega)$. As hyperbolic, u is isolated, i.e. for ε small enough, u is the unique fixed point of R_A in $\overline{B}(u, \varepsilon)$, where $B(u, \varepsilon)$ denotes the ball in $L^2(D)$ of center u and radius ε. Let $i(.,.)$ be the Leray-Schauder Fixed-Point Index, we have then $i(R_A, B(u, \varepsilon)) \neq 0$. Let $H_n : \overline{B}(u, \varepsilon) \times [0, 1] \mapsto L^2(D)$ be the application defined by

$$H_n(x, t) = t R_{A_n}(x) + (1 - t) R_A(x). \tag{15}$$

Suppose that, for n large enough, H_n is a compact homotopy, then

$$i(R_{A_n}, B(u, \varepsilon)) = i(R_A, B(u, \varepsilon)) \neq 0.$$

It implies that there exists $u_n \in SP(\Omega_n) \cap B(u, \varepsilon)$. And by a direct application of Lemma 3.1 we get that the sequence (u_n) converges strongly to u in $L^2(D)$: this would end the proof.

In order to prove that H_n is a compact homotopy, we have to verify that H_n is compact and that $H_n(x, t) \neq x$ for all $(x, t) \in \partial B(u, \varepsilon) \times [0, 1]$. Compactness of H_n follows from compactness of A_n^{-1} and A^{-1}. Suppose that there exists a sequence $(v_k, t_k) \in \partial B(u, \varepsilon) \times [0, 1]$ such that $H_n(v_k, t_k) = v_k$. Note that for a subsequence, after renumbering, we can assume that $H_n(v_n, t_n) = v_n$. Let v be the weak limit in $L^2(D)$ of v_n, and h the weak limit of $f(v_n)$. Since $H_n(v_n, t_n) = v_n$, the sequence (v_n) converges strongly to $v \in \partial B(u, \varepsilon)$ in $L^2(D)$ and as f is lipschitz continuous, $h = f(v)$. On the other hand, by Lemma 3.1, $(R_{A_n}(v_n))_n$ and $(R_A(v_n))_n$ converge strongly in $L^2(D)$ to $A^{-1}(h) = A^{-1}(f(v))$. So, by (15), $v = R_A(v)$. This contradicts the fact that u is the unique fixed point in $\overline{B}(u, \varepsilon)$. $\qquad\square$

Proof of Theorem 3.5. The fact that $u \in SP(\Omega)$ is clear. The two convergences come (see [K, IV, 3.4–3.5]) from Lemma 3.3: the convergence of the resolvent operators $((A_n + \lambda - f'(u_n))^{-1})$ to the resolvent operator $(A + \lambda - f'(u))^{-1}$ in $\mathcal{L}(L^2(D))$. Note that these operators have of course the same eigenvalues and eigenspaces as $((A_n - f'(u_n))^{-1})$, $(A - f'(u))^{-1}$. $\qquad\square$

Remark 3.6. 1) We don't need (C_2) for Theorem 3.4, but it is crucial for the proof of Theorem 3.5. However, (C_2) is of course not a necessary condition for having Theorem 3.5.

We will show now that if the flux in the domain Ω has no non-hyperbolic equilibrium, then the number of equilibria is the same in the limit and the perturbed domains. We begin with a lemma referring to a slightly more general situation.

Lemma 3.7. *Let $u \in SP^+(\Omega) \cup SP^-(\Omega)$. If there exists two sequences (u_n), (v_n) with $u_n \in SP(\Omega_n)$ and $v_n \in SP(\Omega_n)$ converging both to u in $L^2(D)$, then, for n large enough, $u_n = v_n$.*

Proof. Assume that $u_n \neq v_n$ on a set of positive measure. Define $w_n = (u_n - v_n)/\|u_n - v_n\|_{L^2(\Omega_n)}$. Note that by (C_2), $\|w_n\|_{L^2(\Omega)} = \|w_n\|_{L^2(\Omega_n)} = 1$. We will show that

$$w_n \to 0 \quad \text{strongly in } L^2(\Omega),$$

which will obviously be a contradiction.

From the equation solved by $u_n - v_n$ we see that

$$\|u_n - v_n\|_{H^1(\Omega_n)} \leq C_f \|u_n - v_n\|_{L^2(\Omega_n)}. \tag{16}$$

So, $\|u_n - v_n\|_{H^1(\Omega_n)} \to 0$. Note also that $w_n \in H^1(\Omega_n)$ and is solution of

$$\int_{\Omega_n} \nabla w_n \nabla \varphi_n + \int_{\Omega_n} w_n \varphi_n = \int_{\Omega_n} \frac{f(u_n) - f(v_n)}{\|u_n - v_n\|_{L^2(\Omega_n)}} \varphi_n, \ \ \forall \varphi_n \in H^1(\Omega_n),$$

which by (C_2) is the same as

$$\int_\Omega \nabla w_n \nabla \varphi_n + \int_\Omega w_n \varphi_n = \int_\Omega \frac{f(u_n) - f(v_n)}{\|u_n - v_n\|_{L^2(\Omega)}} \varphi_n, \ \ \forall \varphi_n \in H^1(\Omega_n). \quad (17)$$

Taking w_n as test function in (17), we get that $\|w_n\|_{H^1(\Omega_n)}$ is uniformly bounded. So, up to a subsequence, thanks to the Mosco condition (M_1), there exists $w \in H^1(\Omega)$ such that

$$(p_n(\nabla w_n), p_n(w_n)) \xrightarrow{[L^2(D)]^{N+1}} (p(\nabla w), p(w)). \quad (18)$$

We will now prove that w is solution of

$$\int_\Omega \nabla w \nabla \varphi + \int_\Omega w \varphi = \int_\Omega f'(u) w \varphi, \ \ \forall \varphi \in H^1(\Omega). \quad (19)$$

This, since $u \in SP^+(\Omega) \cup SP^-(\Omega)$, gives us $w = 0$. Let $\varphi \in H^1(\Omega)$ and $\varphi_n \in H^1(\Omega_n)$ the sequence given by the Mosco condition (M_2). Passing to the limit in (17), in order to obtain (19), we have to prove that

$$\lim_{n\to\infty} \int_\Omega \frac{f(u_n) - f(v_n)}{\|u_n - v_n\|_{L^2(\Omega_n)}} \varphi_n = \int_\Omega f'(u) w \varphi. \quad (20)$$

Since the injection from $H^1(\Omega)$ into $L^2(\Omega)$ is compact (C_3), the function f can be considered as belonging to $\mathcal{L}(H^1(\Omega), L^2(\Omega))$, and

$$\left\| \frac{f(v_n) - f(u_n)}{\|u_n - v_n\|_{L^2(\Omega_n)}} - f'(u_n) w_n \right\|_{L^2(\Omega)} = \frac{o\|u_n - v_n\|_{H^1(\Omega)}}{\|u_n - v_n\|_{L^2(\Omega_n)}} \quad (21)$$

With (16) we have

$$\frac{o\|u_n - v_n\|_{H^1(\Omega)}}{\|u_n - v_n\|_{L^2(\Omega_n)}} \ \leq \ C_f \frac{o\|u_n - v_n\|_{H^1(\Omega)}}{\|u_n - v_n\|_{H^1(\Omega)}}. \quad (22)$$

And since $\|u_n - v_n\|_{H^1(\Omega)} \to 0$, it follows that

$$\frac{f(v_n) - f(u_n)}{\|u_n - v_n\|_{L^2(\Omega_n)}} - f'(u_n) w_n \longrightarrow 0 \ \text{ in } \ L^2(\Omega). \quad (23)$$

Thus, we are in position for passing to the limit in (17), obtain (19), and conclude that $w = 0$. We got our contradiction: w_n (a subsequence of) converges strongly in $L^2(\Omega)$ to 0 and $\|w_n\|_{L^2(\Omega)} = 1$. $\qquad\square$

Thus, if no non-hyperbolic equilibria exist, the structure of the set of equilibria in Ω_n is very similar to the one in the limit domain.

Theorem 3.8. *Suppose that $SP(\Omega)^0 = \emptyset$. Then for n large enough*

$$\text{card}SP^+(\Omega_n) = \text{card}SP^+(\Omega),$$
$$\text{card}SP^-(\Omega_n) = \text{card}SP^-(\Omega).$$

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Proof. This is an immediate consequence of Theorems 3.4, 3.5 and of Lemma 3.7.
□

Theorem 3.9. *Suppose that $SP(\Omega)^0 = \emptyset$, then*

$$\lim_{n \to \infty} d^H(SP^+(\Omega_n), SP^+(\Omega)) = 0,$$
$$\lim_{n \to \infty} d^H(SP^-(\Omega_n), SP^-(\Omega)) = 0.$$

Here, d^H denotes the Hausdorff distance between sets in $L^2(D)$. The functions are considered, as usual, in $L^2(D)$ by extension by zero.

Proof. The assumption (7), on f implies that $SP(\Omega)$ is bounded in $H^1(\Omega)$ and then compact in $L^2(\Omega)$. So, if $SP(\Omega)$ is hyperbolic, the subset $SP^+(\Omega) \cup SP^-(\Omega)$ is finite. It is then clear, by Theorems 3.4 and 3.5, that

$$\lim_{n \to \infty} \sup_{u \in SP^+(\Omega)} \inf_{v \in SP^+(\Omega_n)} \|u - v\|_{L^2(D)} = 0.$$

On the other hand, let $u_n \in SP^+(\Omega_n)$. As $SP(\Omega_n)$ is uniformly bounded with respect to n in $H^1(\Omega_n)$, using the Mosco conditions, it is easy to see that, up to a subsequence, u_n converge strongly to $u \in SP(\Omega)$ in $L^2(\Omega)$. Since $u_n \in L^\infty(\Omega_n)$, the convergence holds in $L^2(D)$. Using Theorem 3.5 again, we conclude that $u \in SP^+(\Omega)$ and then

$$\lim_{n \to \infty} \sup_{v \in SP^+(\Omega_n)} \inf_{u \in SP^+(\Omega)} \|u - v\|_{L^2(D)} = 0.$$

This ends the proof of the first claim. For the second, we use the same argument. Moreover, with the first Mosco condition, we have $\limsup \lambda(\Omega_n) \leq \lambda(\Omega)$ so if (u_n) is a sequence in $SP^-(\Omega_n)$, each limit of a subsequence is in $SP^-(\Omega) \cup SP^0(\Omega)$, so in $SP^-(\Omega)$. This simplifies the proof in this case. □

4 Convergence of semigroups

We are now interested in the continuity of the parabolic Neumann problems under the perturbation of domain defined above. According to Corollary 2.5, if we restrict the space of initial conditions to $L^\infty(D)$, we can suppose that f is globally lipschitz continuous.

We know that under the condition (G), or (7), for all $u_0 \in L^2(\Omega)$, there exists a unique solution of (6). The map $T_\Omega : \mathbb{R}^+ \times L^2(\Omega) \mapsto L^2(\Omega)$ which to (t, u_0) associates the solution of the equation (6) is a nonlinear semigroup. Let us make this more specific. Take $\beta > 0$ and $T > 0$. We define $L^2_\beta((0,T), L^2(D))$ as the Banach space of functions defined on $(0,T) \times \Omega$, endowed with the norm

$$\Big(\int_0^T e^{-\beta t} \|u(t)\|^2_{L^2(D)} dt \Big)^{\frac{1}{2}}.$$

Let $\mathcal{F} : L^2_\beta((0,T), L^2(D)) \mapsto L^2_\beta((0,T), L^2(D))$ be the map

$$\mathcal{F}(u)(t) = \int_0^t S(t-s) f(u(s)) ds.$$

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We get from (5) by a simple calculation that $\mathcal{F}$ is lipschitz with a constant smaller than $\frac{\text{Lipf}}{\sqrt{\beta+2}}$. Hence, for β large enough, it is strictly contractive, uniformly in T and in Ω.

The solution of (8) is then the unique fixed point in $L^2_\beta((0,T), L^2(D))$ of the map $\mathcal{G}: L^2_\beta((0,T), L^2(D)) \mapsto L^2_\beta((0,T), L^2(D))$

$$\mathcal{G}(u)(t) = S(t)u_0 + \mathcal{F}(u)(t)$$

We define in the same way the map $\mathcal{F}_n$ and $\mathcal{G}_n$ for the semigroup S_n; they are uniformly contractive as well, with respect to T and n.

Theorem 4.1. *Suppose that $(\Omega_n)_{n \in \mathbb{N}}$ converges to Ω in the sense of Mosco. For all $u_0 \in L^\infty(D)$ and all $T > 0$*

$$\lim_{n \to \infty} \|T_n(t)u_0 - T(t)u_0\|_{L^2_\beta((0,T),L^2(D))} = 0 \tag{24}$$

where β is such that $\sqrt{\frac{C_f}{\beta+1}} < 1$.

In order to prove this result, we need the resolvent operator and the linear semigroup continuity.

Lemma 4.2. *Assume that (Ω_n) converge to Ω in the sense of Mosco. For all $h \in L^2(D)$, and all $\lambda \in \mathcal{S}$,*

$$\lim_{n \to \infty} \|(\lambda - A_n)^{-1}(h) - (\lambda - A)^{-1}(h)\|_{L^2(D)} = 0.$$

Proof of Lemma 4.2. We will argue as in the proofs of Lemma 3.1 and Theorem 3.5. Let $u_n = (\lambda - A_n)^{-1}(h)$, by the inequality (3) and (M_1) there exists $u \in H^1(\Omega)$ such that, up to a subsequence,

$$(p_n(\nabla u_n), p_n(u_n)) \xrightarrow{[L^2(D)]^{N+1}} (p(\nabla u), p(u)).$$

Let $\varphi \in H^1(\Omega)$ and let $\varphi_n \in H^1(\Omega_n)$ be given by (M_2). Take them as test functions in the eigenvalue equation for u_n:

$$\int_{\Omega_n} \{\nabla u_n \nabla \varphi_n + u_n \varphi_n - \lambda u_n \varphi_n\} = \int_{\Omega_n} h\varphi_n,$$

for all $n \in \mathbb{N}$. Passing to the limit, we obtain that $u = (\lambda - A)^{-1}(h)$. Using u_n as test function in the above equation, as $\int_{\Omega_n} hu_n \longrightarrow \int_\Omega hu$, we obtain

$$\lim_{n \to \infty} \int_{\Omega_n} \{|\nabla u_n|^2 + |u_n|^2 - \lambda |u_n|^2\} = \int_\Omega \{|\nabla u|^2 + |u|^2 - \lambda |u|^2\}.$$

It follows that

$$\lim_{n \to \infty} \int_D \{|p_n(\nabla u_n) - p(\nabla u)|^2 + (1 - \lambda)|p_n(u_n) - p(u)|^2\} = 0.$$

Since $\lambda \in \mathcal{S}$, if its imaginary part is equal to zero, then $\lambda < 1$. We get

$$\lim_{n \to \infty} \int_D |p_n(u_n) - p(u)|^2 = 0.$$

$\square$

which is the desired result.

Corollary 4.3. *For all compact $K \subset \mathcal{S}$ and $h \in L^2(D)$,*

$$\lim_{n \to \infty} \sup_{\lambda \in K} \|(\lambda - A_n)^{-1}(h) - (\lambda - A)^{-1}(h)\| = 0.$$

Proof of Corollary 4.3. This is a direct consequence of the resolvent identity. Let $r_0 = \min\{|\lambda - 1|, \lambda \in K\}$. Since $1 \notin \mathcal{S}$, we have $r_0 > 0$. Let $(\lambda_n)_{n \in \mathbb{N}}$ be a maximizing sequence. Then, up to a subsequence, it converges to some λ. On the other hand, by the resolvent identity, we have

$$\|(\lambda_n - A_n)^{-1}h - (\lambda_n - A)^{-1}h\|_{L^2(D)}$$
$$\leq \frac{8|\lambda_n - \lambda|}{r_0^2}\|h\| + \|(\lambda - A_n)^{-1}h - (\lambda - A)^{-1}h\|_{L^2(D)}.$$

We apply now Lemma 4.2 and the proof is finished. □

As a consequence of Lemma 4.2, we prove the linear semigroup continuity.

Lemma 4.4. *Suppose that $(\Omega_n)_{n \in \mathbb{N}}$ converges to Ω in the sense of Mosco. For all $u_0 \in L^2(D)$ and all T, δ such that $0 < \delta < T$*

$$\lim_{n \to \infty} \sup_{[\delta, T]} \|S_n(t)u_0 - S(t)u_0\|_{L^2(D)} = 0.$$

Proof of Lemma 4.4. Let Γ be a smooth contour included in $-\mathcal{S}$, i.e. in the region $\{\lambda : 0 \leq |arg(\lambda + 1)| \leq \frac{2\pi}{3}, \lambda \neq -1\}$, and such that for some $R_0 > 0$

$$\lambda \in \Gamma \setminus B(-1, R_0) \implies \lambda = -1 + re^{\pm i\theta},$$

with some fixed $\theta \in (\frac{\pi}{2}, \frac{2\pi}{3})$, and $r \in (R_0, \infty)$. Using (4), we have for all $t > 0$

$$S_n(t)u_0 - S(t)u_0 = \frac{1}{2i\pi} \int_\Gamma e^{\lambda t}[(\lambda + A_n)^{-1}u_0 - (\lambda + A)^{-1}u_0]d\lambda.$$

Using (3), for all $R > R_0$ we obtain the following inequalities, where $r = |\lambda + 1|$:

$$\left\| \frac{1}{2i\pi} \int_{\Gamma \cap \bar{B}(1,R)^c} e^{\lambda t} \left[(\lambda + A_n)^{-1}u_0 - (\lambda + A)^{-1}u_0\right] d\lambda \right\|_{L^2(D)}$$
$$\leq \frac{2}{\pi}\|u_0\|_{L^2(D)} \int_{\Gamma \cap \bar{B}(1,R)^c} \frac{|e^{\lambda t}|}{r} d|\lambda|$$
$$\leq \frac{4}{\pi}\|u_0\|_{L^2(D)} \int_R^{+\infty} \frac{e^{-t+rt\cos\theta}}{R} dr \leq -\frac{4}{\pi}\|u_0\|_{L^2(D)} \left[\frac{e^{-t+Rt\cos\theta}}{Rt\cos\theta}\right],$$

since $\pi > \theta > \frac{\pi}{2}$. Also, for these values of θ,

$$\lim_{R \to +\infty} \sup_{[\delta, T]} \frac{e^{t(-1+R\cos\theta)}}{Rt\cos\theta} = 0.$$

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Take now $\varepsilon > 0$, there exists $R > 0$ such that

$$\sup_{[\delta,T]} \left\| S_n(t)u_0 - S(t)u_0 \right\|_{L^2(D)}$$

$$\leq \frac{1}{2\pi} \int_{\Gamma \cap \bar{B}(1,R)} \left| e^{\lambda T} \right| \left\| (\lambda + A_n)^{-1}u_0 - (\lambda + A)^{-1}u_0 \right\|_{L^2(D)} d\lambda + \varepsilon.$$

The result follows by Corollary 4.3. $\qquad\square$

We are in position to prove Theorem 4.1.

Proof of Theorem 4.1. Since $\mathcal{G}_n$ are uniformly contractive in $L^2_\beta((0,T), L^2(D))$, it is sufficient to prove that for all $p \in \mathbb{N}$, and $\varphi \in L^2_\beta((0,T), L^2(D))$, $\mathcal{G}_n^p(\varphi)$ converges to $\mathcal{G}_n^p(\varphi)$ strongly in $L^2_\beta((0,T), L^2(D))$. Proceeding by recurrence, we have to prove that

if $\varphi_n \to \varphi$ in $L^2_\beta((0,T), L^2(D))$, then $\mathcal{G}_n(\varphi_n) \to \mathcal{G}_n(\varphi)$ in $L^2_\beta((0,T), L^2(D))$.

It is clear, by Lemma 4.4, that $S_n(t)u_0 \to S(t)u_0$ in $L^2_\beta((0,T), L^2(D))$. Hence it is sufficient to prove that $\mathcal{F}_n(\varphi_n) \to \mathcal{F}_n(\varphi)$ in $L^2_\beta((0,T), L^2(D))$. Since $\mathcal{F}_n$ are uniformly lipschitz continuous, we have to prove that for all $\varphi \in L^2_\beta((0,T), L^2(D))$,

$$\mathcal{F}_n(\varphi_n) \longrightarrow \mathcal{F}_n(\varphi) \quad \text{in} \quad L^2_\beta((0,T), L^2(D)).$$

Lemma 4.4 implies that $S_n(t-s)f(\varphi(s)) \to S(t-s)f(\varphi(s))$ in $L^2(D)$ a.e. in $(0,t)$. Since

$$\|S_n(t-s)f(\varphi(s))\|_{L^2(D)} \leq e^{-(t-s)}(C + C_f^K \|\varphi(s)\|_{L^2(D)})$$

with $\varphi(s) \in L^2_\beta((0,T), L^2(D))$, the result follows by Lebesgue's Dominated Convergence. $\qquad\square$

5 Appendix

We present here more details on the numerical simulations, results of which were shown in the Introduction. Recall that we have taken the Allen-Cahn nonlinearity:

$$g(u) = u(1 - u^2),$$

having two stable zeros 1 and -1, and one unstable equal to 0. We apply to our equation (1) the following semi-implicit scheme based on the concave–convex splitting of the free energy introduced by Eyre [E] and explored in view of unconditional stability for the Allen–Cahn and Cahn–Hilliard equations by Vollmayr-Lee and Rutenberg in [VR]:

$$\frac{u - \tilde{u}}{\tau} - \Delta u = (1 - a)u + a\tilde{u} - \tilde{u}^3,$$

where $\tilde{u}$ is the value from the previous time step. In the experiments we have taken $a = 3$, value for which the time-discrete scheme is stable [VR]. We then use the finite element approximation for the space discretization.
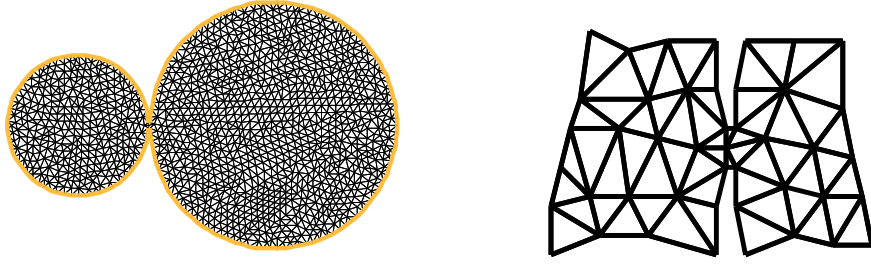
Figure 4: The domain with its FEM grid and a close-up of the grid near the junction of the two subdomains.

The geometry of the domain and the triangulation are shown on Figure 4. This geometry does not follow precisely our model of Figure 1, but is very close to it.- Figure 4 presents a close-up of the junction of the two subdomains. The domain's dimension are:

$$
\begin{array}{rcll}
r_1 & = & \frac{1}{3} * 20 & \text{radius of the left ball,} \\
r_2 & = & \frac{1}{3} * 35 & \text{radius of the right ball,} \\
r_0 & = & \frac{1}{3} & \text{dimension of the 'hole',}
\end{array}
$$

and the numerical parameters:

$$
\begin{array}{rcll}
\tau & = & 0.001 & \text{time–step size,} \\
h_{\max} & = & \frac{1}{3} * 1.75 & \text{diameter of the triangulation,} \\
N & = & 5366 & \text{number of nodes in the triangulation.}
\end{array}
$$

The scaling factor $\kappa = 10/3$ is actually included into the equation: we multiply Δu by $1/\kappa^2$ and work with the equation $u_t - \frac{1}{\kappa^2}\Delta u = g(u)$ and smaller geometrical dimensions.

Results of three simulations are shown in Figure 2.

For an insight into the dynamics of the studied process, let us plot a measure of the rate of change of the function u in time, defined as:

$$
m(t_n) = \int_\Omega \frac{|u^n - u^{n-1}|}{\tau} \tag{25}
$$

Here n is the time step, u^n the numerical solution at time step n, i.e. at $t_n = n\tau$. The measure is more sensitive to changes of u than the rate of change of mass (where mass is defined as $\int_\Omega u$), in particular: $m(t_n) = 0$ implies $u^n \equiv u^{n-1}$ a.e. Figure 5 presents the measure for each of the experiments. Note that logarithmic scale is used for the vertical axes.

One can see that the evolution speed is (i) nearly constant but *increasing* on big intervals of time, and (ii) changing rapidly on some very short intervals. This phenomenon is known to correspond to the evolution on the attractor, (i) following the invariant manifolds and (ii) near the unstable equilibria (see for instance [FH] for the Allen-Cahn case). Our graphs show also a much slower and flatter part by
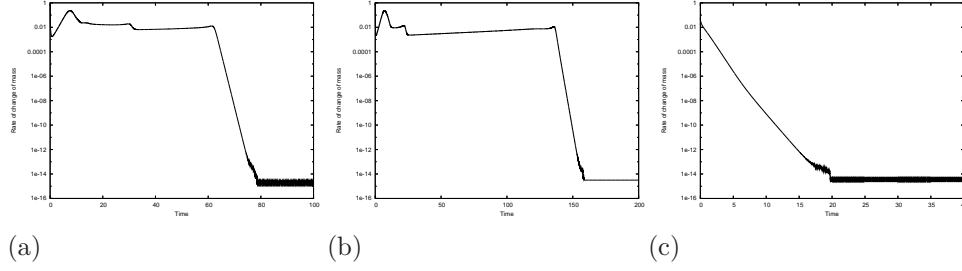
19

(a)　　　　　　　　　　(b)　　　　　　　　　　(c)

Figure 5: The rate of mass change (25) for experiments of Figure 2.

the end of all the experiments, which makes one think there is a particular feature of the final state — this is its stability. This is particularly meaningful when we compare graphs (a) and (b), (c), as the stability of the final state of (a) is clear.

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