POSITION AND SHAPE PARAMETERS OF SECOND ORDER SURFACE ESTIMATED BY POINTS AND INTERVALS

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Introductions

The position, dimensions and shape of shell structures are defined basing on the results of engineering or photogrammetric surveys. These structures are represented by series of points distributed on their external or internal surface, observed from reference points. Number of control points and the way of their distribution depend on the technique of shell observation in use, on accuracy requirements and on local conditions.

The coordinates of control points are determined in 3D space, incidentally the observational model of space network is often reduced to horizontal and height components. Point estimation leading to discovery of unknowns' vector (the coordinates of reference points) is done by least squares method, realising the condition

$$F = \mathbf{v}^T \mathbf{P}\ \mathbf{v} = (\mathbf{w} - \mathbf{Ax})^T\ \mathbf{P}\ (\mathbf{w} - \mathbf{Ax})\ \rightarrow \mathbf{minimum} \tag{1}$$

where

- $\mathbf{w}$ - vector of residuals,
- $\mathbf{A}$ - coefficients' matrix,
- $\mathbf{x}$ - vector of unknowns,
- $\mathbf{P}$ - matrix of weight coefficients in a Markov sense for observed values, i.e. $\mathbf{P} = \mathbf{Cov}(\mathbf{w})^{-1}$.

The basis for making accuracy analysis of calculated surface parameters and their functions is covariance matrix $\mathbf{Cov}(x, y, z)$ for coordinates of observed points. These coordinates are fixed with different accuracy and the degree of accuracy differentiation between individual points depends on the applied measuring technique. For example, in polar method, where coordinates of points P_i are fixed according to dependence:

$$\begin{aligned} x_i &= X_j + d_i \sin\varphi_i \cos A_{ji} \\ y_i &= Y_j + d_i \sin\varphi_i \sin A_{ji} \\ z_i &= Z_j + h_j + d_i \cos\varphi_i \end{aligned} \tag{2}$$

with

- X_j, Y_j, Z_j - station coordinates,
- d_i - slope distance from the station to a point P_i,
- φ_i - zenith angle to the point P_i,
- h_j - height of theodolite,

a priori assumption on the same accuracy of calculated coordinates (what is often done in practice) inescapably leads to distortion of the estimated parameters' values.

Owing to ill-conditioned set of approximation equations, small disturbances in the data cause relatively large disturbances in the solution of the problem. Therefore it is postulated, that for each of the measuring techniques it is good to analyse the accuracy of calculated coordinates of points representing the shell structure.

The covariance matrix $\mathbf{Cov}(x, y, z)$ for coordinates of calculated points will be determined according to the law of covariance propagation. The stochastic model will be accepted under the assumption, that covariances of observations are equal zero and that control points' coordinates X and observations L are independent i.e.

$$\mathbf{Cov}(X, L) = \mathbf{Cov}\begin{pmatrix} X \\ L \end{pmatrix} = \left[\begin{array}{c:c} \mathbf{Cov}(X) & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{Cov}(L) \end{array}\right] \tag{3}$$

while matrix $\mathbf{Cov}(L)$ is a diagonal matrix of a shape

$$\mathbf{Cov}(\hat{L}) = \mathrm{diag}\left\{ V(\alpha), V(\varphi), V(d), V(h) \right\} \tag{4}$$

A matrix of covariances of reference points' coordinates, appearing in the stochastic model, may be approximated according to following models:

Model 1. $\mathbf{Cov}(\hat{X}) = \mathbf{0}$ (5)

- it means assuming coordinates of reference points as errorless.

Model 2. $\mathbf{Cov}(\hat{X}) = \hat{\sigma}^2 \mathbf{E}$ (6)

- it means coordinates equally accurate and uncorrelated ($\hat{\sigma}^2$ - estimator of variance's coefficient, $\mathbf{E}$ - unit matrix).

Model 3. $\mathbf{Cov}(\hat{X}) = \hat{\sigma}^2 \mathbf{D}$ (7)

- it means uncorrelated coordinates of different accuracy ($\mathbf{D}$ - diagonal matrix).

Model 4. $\mathbf{Cov}(\hat{X}) = \hat{\sigma}^2 \mathbf{W}$ (8)

- theoretically a correct model, it takes into account the covariances between the reference points

According to the law of covariance propagation we will write

$$\mathbf{Cov}(x, y, z) = \mathbf{S}^{\mathrm{T}} \mathbf{Cov}(X, L)\, \mathbf{S} \tag{9}$$

where

$\mathbf{S}$ - matrix composed from partial derivatives of functions of (2) type in relation to adequate coordinates of reference points and observations.

A choice of stochastic model depends on definite conditions and required accuracies. It should be noticed that in case of simplified stochastic models (defined by equations (5) (6) or (7)) also numerical calculations may be simplified, because the matrix $\mathbf{Cov}(x, y, z)$ could always be written in a shape:

$$
\mathbf{Cov}(\mathbf{x},\mathbf{y},\mathbf{z}) = \left[\begin{array}{c|c|c|c}
\mathbf{Cov}(x_1, y_1, z_1) & \mathbf{0} & \dots & \mathbf{0} \\ \hline
\mathbf{0} & \mathbf{Cov}(x_2, y_2, z_2) & \dots & \mathbf{0} \\ \hline
\dots & \dots & \dots & \dots \\ \hline
\dots & \dots & \dots & \dots \\ \hline
\mathbf{0} & \mathbf{0} & \dots & \mathbf{Cov}(x_n, y_n, z_n)
\end{array}\right]
$$

In case of redundant observations mathematical model in its stochastic part would be similar to the one given above, while functional part would be formulated through correction equations of a type

$$\boldsymbol{\varepsilon} = \mathbf{B}\,\hat{\mathbf{x}} + \mathbf{w} \tag{10}$$

Estimating parameters of second order surface equation.

General equation of second order surface has the form

$$
\begin{array}{lllllllll}
F(x,y,z) & = & a_{11}x^2 & + & 2a_{12}xy & + & 2a_{13}xz & + & 2a_{14}x & + \\
 & & & & a_{22}y^2 & + & 2a_{23}yz & + & 2a_{24}y & + \\
 & & & & & & a_{33}z^2 & + & 2a_{34}z & + \\
 & & & & & & & & a_{44} & = & 0
\end{array} \tag{11}
$$

The values of parameters a_{ij} are estimated on the grounds of coordinates x,y,z of a specified number of points representing the shell structure.

Equation (11) could be written in the form

$$
\begin{array}{lllllllll}
G(x,y,z) & = & x^2 & + & 2b_{12}xy & + & 2b_{13}xz & + & 2b_{14}x & + \\
 & & & & b_{22}y^2 & + & 2b_{23}yz & + & 2b_{24}y & + \\
 & & & & & & b_{33}z^2 & + & 2b_{34}z & + \\
 & & & & & & & & b_{44} & = & 0
\end{array} \tag{12}
$$

with

$$b_{ij} = \frac{a_{ij}}{a_{11}}; \qquad (b_{11} = 1);$$

In practice number of observed points is always considerably greater than the number of unknown parameters b_{ij}. Therefore instead of the set of equation (12) suitable approximation equations are set together

$$
\begin{array}{lllllllll}
\varepsilon_i & = & x_i^2 & + & 2b_{12}x_iy_i & + & 2b_{13}x_iz_i & + & 2b_{14}x_i & + \\
 & & & & b_{22}y_i^2 & + & 2b_{23}y_iz_i & + & 2b_{24}y_i & + \\
 & & & & & & b_{33}z_i^2 & + & 2b_{34}z_i & + \\
 & & & & & & & & b_{44}
\end{array}
$$

(13)

To increase the accuracy of computations the linear (in relation to unknowns) function (13) could be expanded into series and then its matrix notation is

$$\mathbf{B}\,\mathbf{x} = \mathbf{g} + \varepsilon \tag{14}$$

with

$$\mathbf{B}_{(n,u)} = \begin{bmatrix} 2x_1y_1 & 2x_1z_1 & . & . & . & 1 \\ 2x_2y_1 & 2x_2z_2 & . & . & . & 1 \\ 2x_3y_3 & 2x_3z_3 & . & . & . & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 2x_ny_n & 2x_nz_n & . & . & . & 1 \end{bmatrix}, \quad \mathbf{x}_{(u,1)} = \begin{bmatrix} db_{12} \\ db_{13} \\ db_{14} \\ . \\ . \\ db_{44} \end{bmatrix}, \quad \mathbf{g}_{(n,1)} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ . \\ . \\ g_n \end{bmatrix}, \quad \varepsilon_{(n,1)} = \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ . \\ . \\ \varepsilon_n \end{bmatrix}$$

(n - number of observed points, u - number of unknowns).

$\boldsymbol{\varepsilon}$ is a random vector with covariance matrix $\mathbf{Cov}(\varepsilon)$, which is defined according to dependence

$$\mathbf{Cov}(\varepsilon) = \mathbf{N}^T \mathbf{Cov}(x, y, z)\, \mathbf{N} \qquad (15)$$

where

N - is a matrix composed from partial derivatives of functions of type (12) in relation to the coordinates x,y,z of observed points (it is a matrix of components of vectors normal to the surface),

$\mathbf{Cov}(x, y, z)$ - is covariance matrix for coordinates of points representing the shell.

Matrix **N** is defined through

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_2 & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{N}_m \end{bmatrix}, \quad \mathbf{N}_i = \begin{bmatrix} N_{x_i} = 2\,(x_i + y_i\tilde{b}_{12} + z_i\tilde{b}_{13} + \tilde{b}_{14}) \\ N_{y_i} = 2\,(x_i\tilde{b}_{12} + y_i\tilde{b}_{22} + z_i\tilde{b}_{23} + \tilde{b}_{24}) \\ N_{z_i} = 2\,(x_i\tilde{b}_{31} + y_i\tilde{b}_{32} + z_i\tilde{b}_{33} + \tilde{b}_{34}) \end{bmatrix}$$

The unbiased estimator $\hat{\mathbf{x}}$ of the vector of unknowns will be evaluated with solving generalized linear problem of the least squares method:

$$\boldsymbol{\varepsilon}^T \mathbf{P}\, \boldsymbol{\varepsilon} = \left(\mathbf{g} - \mathbf{B}\hat{\mathbf{x}}\right)^T \mathbf{Cov}(\varepsilon)^{-1} \left(\mathbf{g} - \mathbf{B}\hat{\mathbf{x}}\right) = \mathbf{v}^T \mathbf{P}\, \mathbf{v} \rightarrow \text{ minimum} \qquad (16)$$

The vector $\hat{\mathbf{x}}$ of estimated surface parameters will be evaluated with the help of pseudoinverse matrix

$$\hat{\mathbf{x}} = \mathbf{B}^+\ \mathbf{w} \qquad (17)$$

The empirical value of variance estimator $\hat{\sigma}_o^2$, being the punctual evaluation of the solution (16) is defined generally in accordance with formula

$$\hat{\sigma}_o^2 = \frac{\left\| \mathbf{P}^{-\frac{1}{2}} \left(\mathbf{w} - \mathbf{B}\,\hat{\mathbf{x}}\right) \right\|^2}{n - u} = \frac{\mathbf{v}^T \mathbf{P}\ \mathbf{v}}{n - u}$$

Confidence interval for the variance is based on χ^2 statistic. Using the variance estimator, for (n-u) degrees of freedom and a certain confidence level $(1-\alpha)$ occurs following relationship

$$P\left\{\chi^2\left(\frac{\alpha}{2}, n-u\right) < \frac{(n-u)\hat{\sigma}_o^2}{\sigma_o^2} < \chi^2\left(1-\frac{\alpha}{2}, n-u\right)\right\} = 1-\alpha$$

where

$\chi^2\left(1-\frac{\alpha}{2}, n-u\right)$ and $\chi^2\left(\frac{\alpha}{2}, n-u\right)$ - quantile of an order respectively $\left(1-\frac{\alpha}{2}\right)$ and $\left(\frac{\alpha}{2}\right)$ of χ^2 distribution with (n-u) degrees of freedom.

It should be noticed, that in our case standard deviation $\hat{\sigma}_o$ is only a measure of fitting the mathematical model to observed geometric state of the shell and basing on it we can only conclude about adequacy of the model. However it can not enter into accuracy evaluation of estimated parameters and their functions, because apart from survey errors it includes errors in set-up of the shell and its deformations. Precise separating of these errors is not possible of course. To perform the valuation of accuracy one should a priori estimate the influence of errors scoring from measurements $\hat{\sigma}_{pom}^2$, and this measure accept as the estimator of the unit variance.

According to the above, covariance matrix for the vector of unknowns $\hat{\mathbf{x}}$ is expressed by the formula

$$\mathbf{Cov}(\hat{x}) = \hat{\sigma}_{pom}^2 \mathbf{B}^+ \left(\mathbf{B}^+\right)^T \tag{18}$$

Confidence interval for the unknown surface parameters results from Student's statistic. Using symmetric two-sided intervals we can write

$$\hat{\mathbf{x}} = \pm t\left(1-\frac{\alpha}{2}, n-u\right)\hat{\boldsymbol{\sigma}}_x \tag{19}$$

where

$t\left(1-\frac{\alpha}{2}, n-u\right)$ means quantile of order $\left(1-\frac{\alpha}{2}\right)$ in Student's distribution with (n-u) degrees of freedom, while $\hat{\boldsymbol{\sigma}}_x$ is the estimator of standard deviation for individual unknowns. Components of this vector are computed using matrix (18).

$$\hat{\sigma}_{x_i} = \sqrt{\left[\mathbf{Cov}(\hat{x})\right]_{i,i}} \tag{20}$$

Estimating coordinates of the centre for second order surface.

All the diameters of a central quadric, i.e. the surface for which invariable

$$K = \det\begin{bmatrix} 1 & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \tag{21}$$

is different from zero, criss-cross in one point called the centre of quadric. Using the components of a vector normal to the surface of second order, which coordinates for the point fixing the centre of symmetry will equal zero, we can set following system of equations:

$$\begin{aligned} x_o + b_{12}y_o + b_{13}z_o + b_{14} &= 0 \\ b_{21}x_o + b_{22}y_o + b_{23}z_o + b_{24} &= 0 \\ b_{31}x_o + b_{32}y_o + b_{33}z_o + b_{34} &= 0 \end{aligned} \tag{22}$$

Applying Cramer's formulae, with symbols

$$K_x = \det\begin{bmatrix} -b_{14} & b_{12} & b_{13} \\ -b_{24} & b_{22} & b_{23} \\ -b_{34} & b_{32} & b_{33} \end{bmatrix} \qquad K_y = \det\begin{bmatrix} 1 & -b_{14} & b_{13} \\ b_{21} & -b_{24} & b_{23} \\ b_{31} & -b_{34} & b_{33} \end{bmatrix}$$

$$K_z = \det\begin{bmatrix} 1 & b_{12} & -b_{14} \\ b_{21} & b_{22} & -b_{24} \\ b_{31} & b_{32} & -b_{34} \end{bmatrix}$$

estimators of the coordinates will be calculated from

$$\hat{x}_o = \frac{K_x}{K} \qquad \hat{y}_o = \frac{K_y}{K} \qquad \hat{z}_o = \frac{K_z}{K} \tag{23}$$

Covariance matrix for these estimators is given by

$$\mathbf{Cov}(\hat{x}_o, \hat{y}_o, \hat{z}_o) = \mathbf{S}^T \ \mathbf{Cov}(\hat{\mathbf{x}}) \ \mathbf{S} \tag{24}$$

with

$$\mathbf{S} = \left[\mathbf{S}_{x_o} \ \vdots \ \mathbf{S}_{y_o} \ \vdots \ \mathbf{S}_{z_o}\right]$$

where

$\mathbf{S}_{x_o}, \mathbf{S}_{y_o}, \mathbf{S}_{z_o}$ are vectors of partial derivatives of function (23) in relation to parameters b_{ij} of the surface.

On the grounds of estimated covariance matrix we can fix, at the level $(1-\alpha)$, a confidence ellipsoid for the centre of approximated surface, with semi-axes

$$a_{i_{(1-\alpha)}} = \sqrt{\lambda_i \times \left(\chi^2_{3,1-\alpha}\right)}$$

and corresponding to them normalised direction vectors $\tau(\lambda)_i$.

The latent roots λ_i and attached to them latent vectors s_i result from spectral decomposition of matrix (24), while $\left(\chi^2_{3,1-\alpha}\right)$ is a quantile of order $(1-\alpha)$ in χ^2 distribution with 3 degrees of freedom.

Estimating principal semi-axes of second order surface.

The precise dimensions of principal semi-axes of second order surface one ought to transform function (12) into canonical through appropriate turn and displacement of coordinate system axes.

Directions of principal axes are in line with directions of latent vectors of matrix composed from coefficients of distinctive quadratic form of equation (12)

$$\xi = \begin{bmatrix} 1 & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \tag{25}$$

The latent roots of this matrix result from the condition:

$$\det\begin{bmatrix} 1-\lambda & b_{12} & b_{13} \\ b_{21} & b_{22}-\lambda & b_{23} \\ b_{31} & b_{32} & b_{33}-\lambda \end{bmatrix} = 0$$

Using the calculated latent roots $\lambda_1, \lambda_2, \lambda_3$ we can present the surface equation in the canonical form. If invariable $K \neq 0$ (central surfaces), then

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + \frac{L}{K} = 0 \tag{26}$$

with

$$L = \begin{vmatrix} 1 & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{vmatrix}$$

For $K = 0$, $J \neq 0$, the surface equation can be reduced to the form

$$\lambda_1 x^2 + \lambda_2 y^2 \pm 2\sqrt{\frac{L}{J}}\, z = 0 \tag{27}$$

where

$$J = \begin{vmatrix} 1 & b_{12} \\ b_{21} & b_{22} \end{vmatrix} + \begin{vmatrix} 1 & b_{13} \\ b_{31} & b_{33} \end{vmatrix} + \begin{vmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{vmatrix}$$

Arranging the latent roots in such a way, that in case of an ellipsoid the roots with the same sign would fulfil the condition $|\lambda_1| \leq |\lambda_2| \leq |\lambda_3|$, and in case of hyperboloid and a cone $|\lambda_1| \leq |\lambda_2|$, we are then able to calculate the estimators of principal semi-axes of central surfaces from the formulae

$$\hat{a} = \sqrt{-\frac{W}{\lambda_1}} \quad ; \quad \hat{b} = \sqrt{-\frac{W}{\lambda_2}} \quad ; \quad \hat{c} = \sqrt{-\frac{W}{\lambda_3}} \quad ; \tag{28}$$

(it happens that $a \geq b \geq c$ - for an ellipsoid and $a \geq b$ for hyperboloid and a cone)

with

$$W = \frac{L}{K} = \hat{x}_o b_{14} + \hat{y}_o b_{24} + \hat{z}_o b_{34} + b_{44}$$

The accuracy of determined values of principal semi-axes depends on accuracy of surface parameters, those estimated before. Covariance matrix for the semi-axes could be written as general formula

$$\mathbf{Cov}(\hat{a}, \hat{b}, \hat{c}) = \mathbf{S}^T \ \mathbf{Cov}(\hat{\mathbf{x}}) \ \mathbf{S}$$

(29)

with

$$\mathbf{S} = [\mathbf{S}_a \,|\, \mathbf{S}_b \,|\, \mathbf{S}_c]$$

where $\mathbf{S}_a, \mathbf{S}_b, \mathbf{S}_c$ are the vectors of partial derivatives of function (28) type in relation to parameters b_{ij} of surface, while $\mathbf{Cov}(\hat{\mathbf{x}})$ is covariance matrix of these parameters.

Estimating directions of principal axes for model surface.

With each of main directions in the system 0,x,y,z corresponds one of the latent roots. Direction cosines of principal axes, with $b_{11} = 0$, must fulfil the conditions:

$$\begin{bmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{bmatrix}^T \times \begin{bmatrix} 1-\lambda & b_{12} & b_{13} \\ b_{21} & b_{22}-\lambda & b_{23} \\ b_{31} & b_{32} & b_{33}-\lambda \end{bmatrix} = [\,0 \quad 0 \quad 0\,]$$

(30)

where α, β, γ - angles between the main direction and suitable axes of rectangular coordinate system.

Denoting by M_j appropriate matrix minors (25)

$$M_1 = \begin{vmatrix} b_{12} & b_{13} \\ b_{22}-\lambda & b_{23} \end{vmatrix}; \; M_2 = \begin{vmatrix} 1-\lambda & b_{13} \\ b_{21} & b_{23} \end{vmatrix}; \; M_3 = \begin{vmatrix} 1-\lambda & b_{12} \\ b_{21} & b_{22}-\lambda \end{vmatrix} \qquad (31)$$

the solution of the system (30) could be expressed with the formulae:

$$\begin{aligned} \hat{e}_\alpha &= \cos\alpha = tB_\alpha \\ \hat{e}_\beta &= \cos\beta = tB_\beta \\ \hat{e}_\gamma &= \cos\gamma = tB_\gamma \end{aligned} \qquad (32)$$

where : t is a parameter whereas $B_\alpha = \;\; M_1$; $B_\beta = -M_2$; $B_\gamma = \;\; M_3$.

Under assumption, that at least one the minors is different from zero, exercising the condition

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

we reckon

$$t = \frac{1}{\sqrt{B_\alpha{}^2 + B_\beta{}^2 + B_\gamma{}^2}} = \frac{1}{B}$$

Introducing t to equations (32) we will estimate the positions of semi-axes in relation to accepted coordinate system.

Deflection angle of structure's geometric axis from the plumb line is

$$\hat{\gamma} = \operatorname{arc}\,\cos\frac{B_\gamma}{B}$$

Azimuth of inclination will be calculated according to the formula:

$$\hat{\Phi}_\gamma = \operatorname{arc}\,\operatorname{tg}\frac{B_\beta}{B_\alpha}$$

Covariance matrix for direction cosines will be reckoned after

$$\mathbf{Cov}(\hat{\mathbf{e}}_\alpha, \hat{\mathbf{e}}_\beta, \hat{\mathbf{e}}_\gamma) = \mathbf{S}^T \; \mathbf{Cov}(\hat{\mathbf{x}}) \;\; \mathbf{S}$$

with

$$\mathbf{S} = \left[\mathbf{S}_\alpha \;\vdots\; \mathbf{S}_\beta \;\vdots\; \mathbf{S}_\gamma \right]$$

where $\mathbf{S}_\alpha, \mathbf{S}_\beta, \mathbf{S}_\gamma$ - vectors of partial derivatives of function (12) in relation to parameters of the surface equation.

Calculation of deviations of the actual shell form from the model one.

Solving the set of approximation equations we shall also define a fitting vector v expressing not fulfilment by points representing the shell of approximating surface equation.

Using the components $\mathbf{N}_x, \mathbf{N}_y, \mathbf{N}_z$ of the vector **N** normal the model surface, we can write

$$\left[\mathbf{x}_m \quad \mathbf{y}_m \quad \mathbf{z}_m \right] = \left[\mathbf{x} \quad \mathbf{y} \quad \mathbf{z} \right] + \mathbf{p}\left[\mathbf{N}_x \quad \mathbf{N}_y \quad \mathbf{N}_z \right] \tag{33}$$

or

$$\left[\mathbf{x}_m \;\; \mathbf{y}_m \;\; \mathbf{z}_m \right] = \left[\mathbf{x} \;\; \mathbf{y} \;\; \mathbf{z} \right] + \left[\mathbf{v}_x \;\; \mathbf{v}_y \;\; \mathbf{v}_z \right] \tag{34}$$

where

$\left[\mathbf{x} \quad \mathbf{y} \quad \mathbf{z} \right]$ - coordinate matrix for points representing the actual shell,

$\left[\mathbf{x}_m \quad \mathbf{y}_m \quad \mathbf{z}_m \right]$ - coordinate matrix for points of the model shell,

$\left[\mathbf{v}_x \quad \mathbf{v}_y \quad \mathbf{v}_z \right]$ - matrix of form deviations' components between approximated surface and the actual one,

$\left[\mathbf{N}_x \quad \mathbf{N}_y \quad \mathbf{N}_z \right]$ - matrix of components of the vector normal to approximated surface in observed points $P(x, y, z)$,

p - vector of searched parameters.

Substituting in approximation equations the coordinates of observed points with model surface coordinates given by (33) and the vector of unknowns **x** by its estimator $\hat{\mathbf{x}}$, we will get n independent condition equations, which can be written after modifications as

$$\begin{aligned} & p_i^{\,2}\left[N_{x_i}^2 + b_{22}N_{y_i}^2 + b_{33}N_{z_i}^2 + 2b_{12}N_{x_i}N_{y_i} + 2b_{13}N_{x_i}N_{z_i} + 2b_{23}N_{y_i}N_{z_i} \right] + \\ & p_i \left[N_{x_i}^2 + N_{y_i}^2 + N_{z_i}^2 \right] + \\ & \quad x_i^2 + b_{22}y_i^2 + b_{33}z_i^2 + 2b_{12}x_i y_i + 2b_{13}x_i z_i + 2b_{23}y_i z_i + \\ & \quad 2(b_{14}x_i + b_{24}y_i + b_{34}z_i) + b_{44} \;\; = 0 \end{aligned} \tag{35}$$

Solving n square equations of type (35) we shall define the vector of parameters **p.**

The components v_x, v_y, v_z of distance from observed point P to model shell will be computed from

$$v_x = pN_x \quad ; \qquad v_y = pN_y \quad ; \qquad v_z = pN_z \tag{36}$$

whereas the space distance is given by the formula

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = p\sqrt{N_x^2 + N_y^2 + N_z^2} = p \cdot d \qquad (37)$$

Covariance matrix for deviations of surface's form will be estimated according to

$$\mathbf{Cov}(\mathbf{v}) = \mathbf{S}^T \mathbf{Cov}(\mathbf{x}, \mathbf{y}, \mathbf{z})\, \mathbf{S} \qquad (38)$$

where

$$\mathbf{S} = \begin{bmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial z} \end{bmatrix} = \mathbf{d} \begin{bmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{bmatrix} + \mathbf{p} \begin{bmatrix} \frac{\partial d}{\partial x} \\ \frac{\partial d}{\partial y} \\ \frac{\partial d}{\partial z} \end{bmatrix}$$

Realizing the formula (38) we will get accuracy estimation of calculated form deviations of actual surface from the model one.

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