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A MODIFIED ALGORYTHM OF DETERMINING THE SHAPE OF SHELL OBJECTS USING THE METHOD OF CONICAL INTERSECTION

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ABSTRACT

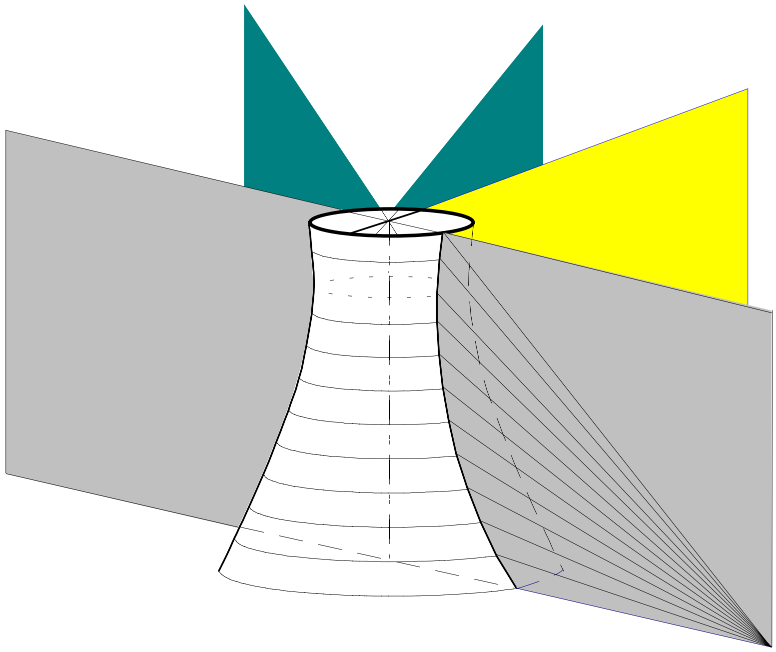
The category of geodetic calculations known as the method of conical intersections could be used for geodetic stocktaking of lines and surfaces. This study presents modified algorithms of the method of conical intersections which allow to improve calculations and to carry out a precise assessment of the accuracy of parameters which are being determined.

PLANE INTERSECTIONS

Conical intersections were formulated and described in the studies [1-3]. Their characteristic feature is the course of the process of approximation of model parameters in two stages. During the first stage one determines the axis of an analysed object on the basis of conical intersection. This allows to determine the deflection of a structure. Then co-ordinates of stations and observation are transformed to a system in which the axis is vertical. Thanks to that one finds the parameters of a shell in the canonical form. Finally, on the basis of determined coefficients of the equation of a mathematical creation one determines deviations of measured points from the model.

The axis of an approximated surface is determined on the basis of plane intersection. A common idea of axis determination is presented on figure 1.

Fig. 1. The axis formed by crossing planes of mean tangency directions



Description of verticality amounts to determination of parameters of axes of an object. In order to that one shall find for each station a plane which comprises the axis of the structure and goes through a point of the station. On the basis of a pencil of planes which is formed in this way one finds averaged parameters of the axis.

Lines which are created by averaged directions to tangency points measured from a given station lie, within the limits of accuracy of a measurement, in one plane. Parameters of that plane will be determined using the least square method, however, matching will make sense when the number of mean directions for every plane is bigger than 2.

An equation of a plane derived from ith station has the following form:

$$A \cdot (X - X_i) + B \cdot (Y - Y_i) + C \cdot (Z - Z_i) = 0$$

where the normal vector of a plane $\vec{N} = [A, B, C]$ and the vector $[X - X_i, Y - Y_i, Z - Z_i]$ are mutually orthogonal while the point (X, Y, Z) is a tangency point and the point $S_i = [X_i, Y_i, Z_i]$ is a station. Values A, B, C will be unknown; they will be determined on the basis of known values $\alpha_{śr}$, $\varphi_{śr}$. Let's form a vector of mean observation direction $\vec{l}$ which will be unitary in a projection onto a plane 0XY $\vec{l} = [cos\,\alpha_{śr}, sin\,\alpha_{śr}, tg\,\varphi_{śr}]$. Vectors $\vec{N}$ and $\vec{l}$ should be mutually perpendicular. This condition results in a following equation:

$$A \cdot cos\,\alpha_{śr} + B \cdot sin\,\alpha_{śr} + C \cdot tg\,\varphi_{śr} = 0\,,$$

from which results an observation equation of the following type:

$$A \cdot cos\,\alpha_{śr} + B \cdot sin\,\alpha_{śr} + C \cdot tg\,\varphi_{śr} = v \qquad (1)$$

It should be stated that the searched vector must be normalised, otherwise there are infinitely many vectors $\vec{N}$ perpendicular to $\vec{l}$. The simplest method to normalise is assuming that one of co-ordinates has the value 1, however only such co-ordinate can be chosen which definitely is different from 0. Because it results from the very assumption that the searched plane should be approximately vertical, $C \approx 0$. Co-ordinates A and B remain, definitely one of them will be different from 0. $\alpha_{śr}$ is the factor which determines which of them shall be assumed as equal to 1.

If $\alpha_{śr} \approx 0^0$ (180^o) $\Rightarrow$ A $\approx$ 0 $\Leftrightarrow$ B $\neq$ 0 it is assumed that B = 1

If $\alpha_{śr} \approx 90^0$ (270^o) $\Rightarrow$ B $\approx$ 0 $\Leftrightarrow$ A $\neq$ 0 it is assumed that A = 1

Let's assume, for example that B $\approx$ *0*, it is assumed that *A = 1*, then the equation (1) shall have the following form:

$$cos\,\alpha_i + B \cdot sin\,\alpha_i + C \cdot tg\,\varphi_i = \upsilon_i$$

when

$i \in$ { 1, 2, ..., n }

n : number of pairs of observations from a given station

When a matrix is formed:

$$\mathbf{A} = \begin{bmatrix} 1 & sin\,\alpha_1 \\ 1 & sin\,\alpha_2 \\ \dots & \dots \\ 1 & sin\,\alpha_n \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} B \\ C \end{bmatrix}, \ \mathbf{L} = \begin{bmatrix} -cos\,\alpha_1 \\ -cos\,\alpha_2 \\ \dots \\ -cos\,\alpha_n \end{bmatrix}$$

one may solve the system **Ax = L** using the least square method. Using this method one will also determine parameters of accuracy assessment, when $\sigma_A = 0$.

In a situation when *B = 1* the whole reasoning is analogous. An equation of intersecting planes for every station results from calculations:

$$\begin{aligned} &A_1 \cdot (X - X_1) + B_1 \cdot (Y - Y_1) + C_1 \cdot (Z - Z_1) = 0 \\ &A_2 \cdot (X - X_2) + B_2 \cdot (Y - Y_2) + C_2 \cdot (Z - Z_2) = 0 \\ &\dots \\ &A_s \cdot (X - X_s) + B_s \cdot (Y - Y_s) + C_s \cdot (Z - Z_s) = 0 \end{aligned} \qquad (2)$$

when

i $\in$ { 1, 2, ..., s }

s : number of all stations

An equation of the axis is determined on the basis of the following equations:

$$X = u_x \cdot t + X_0, \quad Y = u_y \cdot t + Y_0, \quad Z = u_z \cdot t + Z_0$$

when the point $P_0 = (X_0, Y_0, Z_0)$ is any point of the line, however, to make its determination unique we will assume that one of co-ordinates is 0. The axis is approximately vertical, therefore it definitely cuts the horizontal plane, therefore it could be assumed that $Z_0 = 0$.

The vector $\vec{u} = \lfloor u_z, u_y, u_z \rfloor$ is a directional vector of the line. In order to make its determination unique, the vector should be normalised. Because the vector defines the axis which is approximately vertical, it may be assumed that $u_z = 1$.

DETERMINATION OF AN AXIS USING PROFESSOR KADAJ'S METHOD

From among all planes described by the equation (2), one may determine their common edge, that is the axis of an object. In order to do that for every pair of planes one determines an edge of cutting, however, for such planes which intersect at a profitable angle. The best results are obtained when planes are approximately perpendicular. Most often when choosing matching pairs one considers such planes whose dihedral angle belongs to the range <30^0,150^0>. For example a common axis of planes from stations S_1 and S_2 will be determined:

$$X = \frac{B_1 C_2 - B_2 C_1}{A_1 B_2 - A_2 B_1} Z + \frac{B_2 \left(A_1 X_1 + B_1 Y_1 + C_1 Z_i\right) - B_1 \left(A_2 X_2 + B_2 Y_2 + C_2 Z_2\right)}{A_1 B_2 - A_2 B_1}$$
$$Y = \frac{C_1 A_2 - C_2 A_1}{A_1 B_2 - A_2 B_1} Z + \frac{A_1 \left(A_2 X_2 + B_2 Y_2 + C_2 Z_2\right) - A_2 \left(A_1 X_1 + B_1 Y_1 + C_1 Z_i\right)}{A_1 B_2 - A_2 B_1}$$

after a comparison with a parametric equation of the axis we obtain the searched parameters:

$$X_0 = \frac{B_2 \left(A_1 X_1 + B_1 Y_1 + C_1 Z_i\right) - B_1 \left(A_2 X_2 + B_2 Y_2 + C_2 Z_2\right)}{A_1 B_2 - A_2 B_1}$$
$$Y_0 = \frac{A_1 \left(A_2 X_2 + B_2 Y_2 + C_2 Z_2\right) - A_2 \left(A_1 X_1 + B_1 Y_1 + C_1 Z_i\right)}{A_1 B_2 - A_2 B_1}$$

$$u_x = \frac{B_1 C_2 - B_2 C_1}{A_1 B_2 - A_2 B_1}$$
$$u_y = \frac{C_1 A_2 - C_2 A_1}{A_1 B_2 - A_2 B_1}$$

Then the determined parameters of an axis from particular pairs of planes are averaged to obtain in the end a single equation of the axis. This may be done in two ways. First, one may assume the average of values X_0, Y_0, u_x, u_y, or a weighted average, however, then one should know mean errors for every independently determined set of values X_0, Y_0, u_x, u_y. In this study they will be defined as errors of the function $F(A_i,B_i,C_i,A_j,B_j,C_j)$ having previously calculated errors of values A, B, C.

$$\sigma_F = \sqrt{f^T Cov(A_i B_i C_i A_j B_j C_j) f}$$

where

$$f^T = \begin{bmatrix} \frac{\partial F}{\partial A_i} & \frac{\partial F}{\partial B_i} & \frac{\partial F}{\partial C_i} & \frac{\partial F}{\partial A_j} & \frac{\partial F}{\partial B_j} & \frac{\partial F}{\partial C_j} \end{bmatrix}$$

$$Cov(A_i B_i C_i A_j B_j C_j) = \begin{bmatrix} Cov(A_i B_i C_i) & \underline{0} \\ \underline{0} & Cov(A_j B_j C_j) \end{bmatrix}$$

Agreeing of results from particular pairs of planes causes that difficulty that every plane is used to determine a line many limes. Therefore, results depend on each other which is not taken into account by the method.

AN EDGE ACCORDING TO A MODIFIED ALGORITHM

The method proposed by the authors is an exact method which requires a smaller number of calculations and results in more accurate results. It is not necessary to choose planes because all will be taken into account in the process of compensation and only once.

The first stage of calculation involves normalisation of planes which will make it possible to compare equations formed later:

$$\overline{A_i} = \frac{A_i}{\sqrt{A_i^2 + B_i^2 + C_i^2}}, \quad \overline{B_i} = \frac{B_i}{\sqrt{A_i^2 + B_i^2 + C_i^2}}, \quad \overline{C_i} = \frac{C_i}{\sqrt{A_i^2 + B_i^2 + C_i^2}}$$

After taking into account unit normal vectors in an equation (2) and moving to equations of corrections you will get:

$$\begin{aligned} &\overline{A}_1 \cdot (X - X_1) + \overline{B}_1 \cdot (Y - Y_1) + \overline{C}_1 \cdot (Z - Z_1) = v_1 \\ &\overline{A}_2 \cdot (X - X_2) + \overline{B}_2 \cdot (Y - Y_2) + \overline{C}_2 \cdot (Z - Z_2) = v_2 \\ &\ldots \\ &\overline{A}_s \cdot (X - X_s) + \overline{B}_s \cdot (Y - Y_s) + \overline{C}_s \cdot (Z - Z_s) = v_s \end{aligned} \tag{3}$$

The gist of the method involves cutting of a pencil of planes with an additional plane $Z = 0$. In this way planes will be changed into lines and the whole problem will amount to finding a common point for all lines. When you put $Z = 0$ to the equation (3) you will get:

$$
\begin{aligned}
&\overline{A}_1 \cdot (X - X_1) + \overline{B}_1 \cdot (Y - Y_1) - \overline{C}_1 \cdot Z_1 = v_1 \\
&\overline{A}_2 \cdot (X - X_2) + \overline{B}_2 \cdot (Y - Y_2) - \overline{C}_2 \cdot Z_2 = v_2 \\
&\ldots \\
&\overline{A}_s \cdot (X - X_s) + \overline{B}_s \cdot (Y - Y_s) - \overline{C}_s \cdot Z_s = v_s
\end{aligned}
$$

The above system of equations may be presented as a matrix **AX=L**, where

$$
\mathbf{A} = \begin{bmatrix} \overline{A}_1 & \overline{B}_1 \\ \overline{A}_2 & \overline{B}_2 \\ \ldots & \ldots \\ \overline{A}_s & \overline{B}_s \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} X_0 \\ Y_0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} \overline{C}_1 Z_1 + \overline{A}_1 X_1 + \overline{B}_1 Y_1 \\ \overline{C}_2 Z_2 + \overline{A}_2 X_2 + \overline{B}_2 Y_2 \\ \ldots \\ \overline{C}_s Z_s + \overline{A}_s X_s + \overline{B}_s Y_s \end{bmatrix}
$$

Solving of the system using the least square method shall allow to determine an estimator of the searched point of the line X_0, Y_0 and its assessment of accuracy in a form of a covariance matrix **Cov** (X_0, Y_0).

Similarly to co-ordinates X_0, Y_0 you will find co-ordinates of a directional vector of the line u_x, u_y. To do this in the equation (3) the following substitution will be done $Z = u_z = 1$, getting:

$$
\begin{aligned}
&\overline{A}_1 \cdot (X - X_1) + \overline{B}_1 \cdot (Y - Y_1) + \overline{C}_1 \cdot (1 - Z_1) = v_1 \\
&\overline{A}_2 \cdot (X - X_2) + \overline{B}_2 \cdot (Y - Y_2) + \overline{C}_2 \cdot (1 - Z_2) = v_2 \\
&\ldots \\
&\overline{A}_s \cdot (X - X_s) + \overline{B}_s \cdot (Y - Y_s) + \overline{C}_s \cdot (1 - Z_s) = v_s
\end{aligned}
$$

This system may be written in the following form:

$$
\mathbf{A} = \begin{bmatrix} \overline{A}_1 & \overline{B}_1 \\ \overline{A}_2 & \overline{B}_2 \\ \ldots & \ldots \\ \overline{A}_s & \overline{B}_s \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} -\overline{C}_1 \cdot (1 - Z_1) + \overline{A}_1 X_1 + \overline{B}_1 Y_1 \\ -\overline{C}_2 \cdot (1 - Z_2) + \overline{A}_2 X_2 + \overline{B}_2 Y_2 \\ \ldots \\ -\overline{C}_s \cdot (1 - Z_s) + \overline{A}_s X_s + \overline{B}_s Y_s \end{bmatrix}
$$

A solution made using the least square method will allow to determine the point X_1, Y_1 and a covariance matrix **Cov** (X_1, Y_1).

Finally, co-ordinates of the searched vector $\vec{\mathbf{u}}$ will be calculated in the following way:

$$
\begin{aligned}
u_x &= (X_1 - X_0) \pm \sqrt{\mathbf{Cov}\left(\hat{X}_0, \hat{Y}_0,\right)_{1,1} + \mathbf{Cov}\left(\hat{X}_1, \hat{Y}_1,\right)_{1,1}} \\
u_y &= (Y_1 - Y_0) \pm \sqrt{\mathbf{Cov}\left(\hat{X}_0, \hat{Y}_0,\right)_{2,2} + \mathbf{Cov}\left(\hat{X}_1, \hat{Y}_1,\right)_{2,2}}
\end{aligned}
$$

MULTIPLE PLANE INTERSECTIONS

This method allows for taking into account all observations and simultaneous approximation of axis parameters. It is assumed that the axis will be determined as a line. Variants are also possible, e.g. approximation of an axis with a polynomial, however they will not be discussed in this study.

Let's make the point $P_0 = (x_o, y_o, z_o)$ a searched point of the line p and the vector $\vec{u} = [u_x, u_y, u_z]$ a directional vector of the line p. Let's form a vector $\overrightarrow{P_o S_i} = [x_i - x_o, y_i - y_o, z_i - z_o]$ and a vector of observation direction j

from the station S_i $\overrightarrow{e_{ij}} = [u_{ij}, v_{ij}, w_{ij}]$. Vectors $\vec{u}, \overrightarrow{P_o S_i}, \overrightarrow{e_{ij}}$ lie on one plane, therefore a scalar triple product of those vectors is equal to zero:

$$\left(\vec{u} \times \overrightarrow{P_o S_i}\right) \cdot \overrightarrow{e_{ij}} = 0$$

After substitution of co-ordinates:

$$\begin{vmatrix} u_x & u_y & u_z \\ x_i - x_0 & y_i - y_0 & z_i - z_0 \\ u_{ij} & v_{ij} & w_{ij} \end{vmatrix} = 0$$

On the basis of the above relation an observation equation is formed:

$$F_{ij} = \begin{vmatrix} u_x & u_y & u_z \\ x_i - x_0 & y_i - y_0 & z_i - z_0 \\ u_{ij} & v_{ij} & w_{ij} \end{vmatrix} = \delta_{ij} \tag{4}$$

where

$i \in \{ 1, 2, ..., s \}$
$j \in \{ 1, 2, ..., r_i \}$
s : number of stations
r_i : number of observations from the station i

The equation (4) has six unknowns: x_0, y_0, z_0, u_x, u_y, u_z , and four of them are independent. The point $P_0 = (x_i, y_i, z_i)$ is any point of the line and the vector $\vec{U} = [u_x, u_y, u_z]$ can be normalised in any way. The line p is approximately vertical, so it crosses any horizontal plane (e.g. a plane with the height $z = 0$), therefore, it could be assumed that $z_0 = 0$. Also from the condition of approximated verticality of the axis it is determined that $u_z = 1$. When these relations are taken into account, the equation (4) shall look in the following way:

$$\begin{aligned} F_{ij} = u_x (z_i v_{ij} - y_i w_{ij}) + u_y (x_i w_{ij} - z_i u_{ij}) + y_i u_{ij} - x_i v_{ij} + \\ - y_o u_{ij} + x_o v_{ij} + (u_x y_0 - u_y x_o) w_{ij} = \delta_{ij} \end{aligned}$$

Before the beginning of calculations the function F_{ij} should be linearised

$$F_{ij} = F_{ij0} + dF_{ij} = F_{ij0} + \frac{\partial F_{ij}}{\partial u_x} du_x + \frac{\partial F_{ij}}{\partial u_y} du_y + \frac{\partial F_{ij}}{\partial x_0} dx_0 + \frac{\partial F_{ij}}{\partial y_0} dy_0$$

$$\frac{\partial F_{ij}}{\partial u_x} = w_{ij} (y_i - y_0) - v_{ij} (z_i - z_0)$$

$$\frac{\partial F_{ij}}{\partial u_y} = u_{ij} (z_i - z_0) - w_{ij} (x_i - x_0)$$

$$\frac{\partial F_{ij}}{\partial x_0} = u_y w_{ij} - u_z v_{ij}$$

$$\frac{\partial F_{ij}}{\partial y_0} = u_z u_{ij} - u_x w_{ij}$$

An approximate value of the function will be calculated for approximate unknowns

$$F_{ij0} = \tilde{u}_x \left(z_i v_{ij} - y_i w_{ij}\right) + \tilde{u}_y \left(x_i w_{ij} - z_i u_{ij}\right) + y_i u_{ij} - \tilde{x}_i v_{ij} + \\ - \tilde{y}_o u_{ij} + \tilde{x}_o v_{ij} + \left(\tilde{u}_x \tilde{y}_0 - \tilde{u}_y \tilde{x}_o\right) w_{ij}$$

Because the axis is approximately vertical:

$$\tilde{u}_x = \tilde{u}_y = 0$$

Approximated co-ordinates $\tilde{x}_0, \tilde{y}_0$ will be determined from a forward intersection:

$$\tilde{x}_0 = \frac{y_2 - y_1 + x_1\, tg\,\alpha_1 - x_2\, tg\,\alpha_2}{tg\,\alpha_1 - tg\,\alpha_2}, \quad \tilde{y}_0 = \frac{x_2 - x_1 + y_1\, ctg\,\alpha_1 - y_2\, ctg\,\alpha_2}{ctg\,\alpha_1 - ctg\,\alpha_2}$$

Finally, a matrix system of linear equations **Ax = L** can be built, where:

$$\mathbf{A} = \begin{bmatrix} \frac{\partial F_{11}}{\partial u_x} & \frac{\partial F_{11}}{\partial u_y} & \frac{\partial F_{11}}{\partial x_0} & \frac{\partial F_{11}}{\partial y_0} \\ \frac{\partial F_{12}}{\partial u_x} & \frac{\partial F_{12}}{\partial u_y} & \frac{\partial F_{12}}{\partial x_0} & \frac{\partial F_{12}}{\partial y_0} \\ \dots & \dots & \dots & \dots \\ \frac{\partial F_{sr}}{\partial u_x} & \frac{\partial F_{sr}}{\partial u_y} & \frac{\partial F_{sr}}{\partial x_0} & \frac{\partial F_{sr}}{\partial y_0} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} du_x \\ du_y \\ dx_0 \\ dy_0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} -F_{110} \\ -F_{120} \\ \dots \\ -F_{sr0} \end{bmatrix}$$

In result of solving of this system using the least square method, estimators of unknowns and a covariance matrix are obtained for estimated parameters.

Deflection κ and the azimuth of deflection ϑ are determined according to the relation:

$$\kappa = arctg\left(\frac{\sqrt{u_x^{\,2} + u_y^{\,2}}}{u_z}\right), \quad \vartheta = arctg\left(\frac{u_y}{u_x}\right).$$

An analysis of accuracy of deflection and of the azimuth of deflection can be carried out on the basis of the right of determination of variations for correlated values. Finally

$$\sigma_\kappa^{\,2} = \frac{1}{(u_x^{\,2} + u_y^{\,2})(1 + u_x^{\,2} + u_y^{\,2})^2} \begin{bmatrix} u_x \\ u_y \end{bmatrix}^T \begin{bmatrix} V(u_x) & cov(u_x u_y) \\ cov(u_x u_y) & V(u_y) \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

$$\sigma_\vartheta^{\,2} = \frac{1}{(u_x^{\,2} + u_y^{\,2})^2} \begin{bmatrix} -u_y \\ u_x \end{bmatrix}^T \begin{bmatrix} V(u_x) & cov(u_x u_y) \\ cov(u_x u_y) & V(u_y) \end{bmatrix} \begin{bmatrix} -u_y \\ u_x \end{bmatrix}$$

APPROXIMATION OF PARAMETERS

In effect of realisation of algorithms presented here one can determine deflection of the axis κ and an azimuth of deflection ϑ and the point in which the axis cuts through the horizontal axis.

Transformation

The next stage of calculation is determination of such a co-ordinate system where the axis of an object is vertical. In order to do that one shall form a transformation matrix and transform co-ordinates of stations and observations.

Forming of a transformation matrix K

$$K = A_{\upsilon} A_{-\kappa} A_{-\upsilon}$$

where

A_{υ}: a matrix of a revolution around an axis by angle υ

$A_{-\kappa}$: a matrix of a revolution around an axis by angle $-\kappa$

$A_{-\upsilon}$: a matrix of a revolution around an axis by angle $-\upsilon$

$$K = \begin{bmatrix} cos\,\vartheta & -sin\,\vartheta & 0 \\ sin\,\vartheta & cos\,\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} cos\,\kappa & 0 & -sin\,\kappa \\ 0 & 1 & 0 \\ sin\,\kappa & 0 & cos\,\kappa \end{bmatrix} \begin{bmatrix} cos\,\vartheta & sin\,\vartheta & 0 \\ -sin\,\vartheta & cos\,\vartheta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} cos^2\,\vartheta(cos\,\kappa - 1) + 1 & sin\,\vartheta\, cos\,\vartheta(cos\,\kappa - 1) & -sin\,\kappa\, cos\,\vartheta \\ sin\,\vartheta\, cos\,\vartheta(cos\,\kappa - 1) & sin^2\,\vartheta(cos\,\kappa - 1) + 1 & -sin\,\kappa\, sin\,\vartheta \\ sin\,\kappa\, cos\,\vartheta & sin\,\kappa\, sin\,\vartheta & cos\,\kappa \end{bmatrix}$$

Transformation of co-ordinates of stations S_i

$$\begin{bmatrix} \bar{x}_i \\ \bar{y}_i \\ \bar{z}_i \end{bmatrix} = K\left(\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \\ 0 \end{bmatrix}\right)$$

When the axis of an object is vertical, the matrix K is unitary, therefore transformation of co-ordinates amounts to translation by a vector - $[x_0 \quad y_0 \quad 0]$.

A vector transformation of observation e_{ij}

$$\overrightarrow{e_{ij}} = [cos\,\alpha_{ij}, sin\,\alpha_{ij}, tg\,\varphi_{ij}]$$

$$\begin{bmatrix} L_{ij} \\ M_{ij} \\ N_{ij} \end{bmatrix} = K \begin{bmatrix} cos\,\alpha_{ij} \\ sin\,\alpha_{ji} \\ tg\,\varphi_{ij} \end{bmatrix}$$

$$\bar{a}_{ij} = arctg \frac{M_{ij}}{L_{ij}}$$

$$\bar{\varphi}_{ij} = arctg \frac{N_{ij}}{\sqrt{{L_{ij}}^2 + {M_{ij}}^2}}$$

When the axis of an object is vertical, observations do not change. Transformed values form observation sets $O_{ij} = \{\bar{x}_i, \bar{y}_i, \bar{z}_i, \bar{\alpha}_{ij}, \bar{\varphi}_{ij}\}$ where: i ∈ { 1, 2, ..., s }, j∈ {1,2,..., r_i}.

Approximation of parameters can be done using two methods described below.

Method 1
After a transformation of a system of equations the axis of a hyperboloid in a new system is vertical and the surface can be described by a canonical equation

$$f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{(z-z_0)^2}{c^2} - 1 = 0$$

where a, b, c are geometric parameters of a shell and z_0 determines the height of the centre of symmetry in a co-ordinate system.

Let's make the point $P = (x, y, z)$ a point of tangency of the axis of an aiming line from a station $S_i = (x_i, y_i, z_i)$ and a model surface of a hyperboloid.

Let's form a vector $\vec{L} = [x - x_i, y - y_i, z - z_i]$ which lies on the axis of the aiming line and a normal vector to the surface $\vec{N} = [f_x, f_y, f_z]$ in the point P:

$$f_x = \frac{\partial f}{\partial x} = \frac{2x}{a^2}, \quad f_y = \frac{\partial f}{\partial y} = \frac{2y}{b^2}, \quad f_z = \frac{\partial f}{\partial z} = -\frac{2(z-z_0)}{c^2}$$

Vectors $\vec{L}, \vec{N}$ are perpendicular, therefore:

$$\vec{L} \cdot \vec{N} = (x - \bar{x}_i)\frac{x}{a^2} + (y - \bar{y}_i)\frac{y}{b^2} - (z - \bar{z}_i)\frac{z - z_0}{c^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{(z-z_0)^2}{c^2} - \frac{x\bar{x}_i}{a^2} - \frac{y\bar{y}_i}{b^2} + \frac{(z-z_0)(\bar{z}_i - z_0)}{c^2} = 0$$

$$\frac{x\bar{x}_i}{a^2} + \frac{y\bar{y}_i}{b^2} - \frac{(z-z_0)(\bar{z}_i - z_0)}{c^2} = 1$$

The condition which describes aiming axes in combination with observations and known co-ordinates of stations is as follows:

$$x = l\cos\bar{\alpha} + \bar{x}_i, \quad y = l\sin\bar{\alpha} + \bar{y}_i, \quad z = l\,tg\,\bar{\alpha} + \bar{z}_i$$

After putting the above equations in a canonical equation of a hyperboloid and considering the condition for a rotary hyperboloid, that is the condition $a = b$, the following will be obtained:

$$\frac{(l\cos\bar{\alpha} + \bar{x}_i)^2 + (l\sin\bar{\alpha} + \bar{y}_i)^2}{a^2} - \frac{(l\,tg\,\bar{\varphi} + \bar{z}_i - z_0)^2}{c^2} = 1$$
$$\frac{(l\cos\bar{\alpha} + \bar{x}_i)\bar{x}_i + (l\sin\bar{\alpha} + \bar{y}_i)\bar{y}_i}{a^2} - \frac{(l\,tg\,\bar{\varphi} + \bar{z}_i - z_0)(\bar{z}_i - z_0)}{c^2} = 1$$

From the above equations, when l is eliminated, the equation which describes the surface of a hyperboloid and observations will be obtained.

$$\begin{aligned}&\frac{a^2}{c^2}z_0^2 + a^2 + \frac{2a^2 z_0}{c^2}[\bar{z}_i - tg\,\bar{\varphi}(\bar{x}_i\cos\bar{\alpha} - \bar{y}_i\sin\bar{\alpha})] - \frac{a^4}{c^2}tg^2\,\bar{\varphi} + \\ &+\frac{a^2}{c^2}\left\{[\bar{z}_i - tg\,\bar{\varphi}(\bar{x}_i\cos\bar{\alpha} - \bar{y}_i\sin\bar{\alpha})]^2 + tg^2\,\bar{\varphi}(\bar{x}_i\cos\bar{\alpha} - \bar{y}_i\sin\bar{\alpha})^2\right\} + \\ &-(\bar{x}_i\cos\bar{\alpha} - \bar{y}_i\sin\bar{\alpha})^2 = 0\end{aligned} \tag{5}$$

Let's introduce auxiliary variables:

$$\xi = \frac{a^2}{c^2} z_o{}^2 + a^2, \quad \eta = \frac{2a^2 z_0}{c^2}, \quad \zeta = \frac{a^2}{c^2} \tag{6}$$

Coefficients which are a function of observed elements

$$\begin{aligned} F_{ij} &= \bar{z}_i - tg\,\overline{\varphi}_{ij}\left(\bar{x}_i \cos\overline{\alpha}_{ij} + \bar{y}_{ij} \sin\overline{\alpha}_{ij}\right) \\ G_{ij} &= tg^2\,\overline{\varphi}_{ij} \\ H_{ij} &= \left(\bar{x}_i \cos\overline{\alpha}_{ij} - \bar{y}_{ij} \sin\overline{\alpha}_{ij}\right)^2 \end{aligned} \tag{7}$$

After taking into account (6) and (7), you will get a simple notation of the equation (5)

$$\Phi = \xi + \eta F_{ij} + \zeta\left(F_{ij}{}^2 + G_{ij} H_{ij}\right) + \left(\frac{\eta^2}{4} - \xi\zeta\right) G_{ij} - H_{ij} = 0$$

Observation equations will have the following form:

$$\xi + \eta F_{ij} + \zeta\left(F_{ij}{}^2 + G_{ij} H_{ij}\right) + \left(\frac{\eta^2}{4} - \xi\zeta\right) G_{ij} - H_{ij} = \delta_{ij}$$

The function Φ is not a linear function of searched parameters – before beginning of further calculations one stall carry out linearisation

$$\Phi = \Phi_0 + d\Phi = \Phi_0 + \frac{\partial\Phi}{\partial\xi} d\xi + \frac{\partial\Phi}{\partial\eta} d\eta_y + \frac{\partial\Phi}{\partial\xi} \zeta d\zeta$$

$$\begin{aligned} &\xi_0 + \eta_0 F_{ij} + \zeta_0\left(F_{ij}{}^2 + G_{ij} H_{ij}\right) + \left(\frac{\eta_0{}^2}{4} - \xi_0\zeta_0\right) G_{ij} - H_{ij} + \\ &+ \left(1 - \zeta_0 G_{ij}\right) d\xi + \left(F_{ij} + \frac{\eta_0}{2} G_{ij}\right) d\eta + \left(F_{ij}{}^2 + G_{ij} H_{ij} - \xi_0 G_{ij}\right) d\zeta = \delta_{ij} \end{aligned} \tag{8}$$

Approximate parameters can be determined by introduction to the equation (8) of an auxiliary unknown

$$\lambda_0 = \frac{\eta_0{}^2}{4} - \xi_0\zeta_0$$

Approximate unknowns will be obtained from a solution of a system of linear equations

$$\xi_0 + \eta_0 F_{ij} + \zeta_0\left(F_{ij}{}^2 + G_{ij} H_{ij}\right) + \lambda_0 G_{ij} - H_{ij} = 0$$

which can be written in a form of a matrix:

$$\mathbf{A} = \begin{bmatrix} \frac{\partial\Phi_{11}}{\partial\xi} & \frac{\partial\Phi_{11}}{\partial\eta} & \frac{\partial\Phi_{11}}{\partial\zeta} \\ \frac{\partial\Phi_{12}}{\partial\xi} & \frac{\partial\Phi_{12}}{\partial\eta} & \frac{\partial\Phi_{12}}{\partial\zeta} \\ .. & ... & ... \\ \frac{\partial\Phi_{sr_i}}{\partial\xi} & \frac{\partial\Phi_{sr_i}}{\partial\eta} & \frac{\partial\Phi_{sr_i}}{\partial\zeta} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} -\Phi_{110} \\ -\Phi_{120} \\ ... \\ -\Phi_{sr0} \end{bmatrix}$$

Finally:

$$\xi = (\xi_0 + d\xi) \pm \sqrt{Cov\left(\hat{X}\right)_{1,1}}$$
$$\eta = (\eta_0 + d\eta) \pm \sqrt{Cov\left(\hat{X}\right)_{2,2}}$$
$$\zeta = (\zeta_0 + d\zeta) \pm \sqrt{Cov\left(\hat{X}\right)_{3,3}}$$

$$a = \sqrt{\xi - \frac{\eta^2}{4\zeta}}, \quad c = \sqrt{\frac{\xi}{\zeta} - \frac{\eta^2}{4\zeta^2}}, \quad z_0 = \frac{\eta}{2\zeta}$$

Method 2

The equation (5) without introduction of auxiliaries. Unknown parameters a, c, z_0 will be determined after the following transformations:

$$\Phi = a^2\left[z_0 - \bar{z}_i + tg(\bar{x}_i\, cos\,\bar{\alpha} + \bar{y}_i\, sin\,\bar{\alpha})\right]^2 + \\ + \left[a^2 - (\bar{x}_i\, cos\,\bar{\alpha} - \bar{y}_i\, sin\,\bar{\alpha})^2\right] \cdot \left(c^2 - a^2\, tg^2\,\bar{\varphi}\right) = 0$$

$$a^2\left[z_0 - \bar{z}_i + tg(\bar{x}_i\, cos\,\bar{\alpha} + \bar{y}_i\, sin\,\bar{\alpha})\right]^2 + \\ + \left[a^2 - (\bar{x}_i\, cos\,\bar{\alpha} - \bar{y}_i\, sin\,\bar{\alpha})^2\right] \cdot \left(c^2 - a^2\, tg^2\,\bar{\varphi}\right) = \delta_{ij}$$

$$\Phi = \Phi_0 + d\Phi = \Phi_0 + \frac{\partial\Phi}{\partial a}da + \frac{\partial\Phi}{\partial c}dc + \frac{\partial\Phi}{\partial z_0}dz_0$$

$$\frac{\partial\Phi_{ij}}{\partial a} = 2\tilde{a}\left\{\left[z_0 - \bar{z}_i + tg(\bar{x}_i\, cos\,\bar{\alpha}_{ij} + \bar{y}_i\, sin\,\bar{\alpha}_{ij})\right]^2 + \\ + \left(\tilde{c}^2 - \tilde{a}^2\, tg^2\,\bar{\varphi}_{ij}\right) - tg^2\,\bar{\varphi}_{ij} \cdot \left[\tilde{a}^2 - (\bar{x}_i\, cos\,\bar{\alpha}_{ij} - \bar{y}_i\, sin\,\bar{\alpha}_{ij})^2\right]\right\}$$

$$\frac{\partial\Phi_{ij}}{\partial c} = 2\tilde{c}\left[\tilde{a}^2 - (\bar{x}_i\, cos\,\bar{\alpha}_{ij} - \bar{y}_i\, sin\,\bar{\alpha}_{ij})^2\right]$$

$$\frac{\partial\Phi_{ij}}{\partial z_0} = 2\tilde{a}^2\left[\tilde{z}_0 - \bar{z}_i + tg(\bar{x}_i\, cos\,\bar{\alpha}_{ij} + \bar{y}_i\, sin\,\bar{\alpha}_{ij})\right]$$

$$\Phi_{ij0} = \tilde{a}^2\left[\tilde{z}_0 - \bar{z}_i + tg(\bar{x}_i\, cos\,\bar{\alpha}_{ij} - \bar{y}_i\, sin\,\bar{\alpha}_{ij})\right]^2 + \\ + \left[\tilde{a}^2 - (\bar{x}_i\, cos\,\bar{\alpha}_{ij} - \bar{y}_i\, sin\,\bar{\alpha}_{ij})^2\right] \cdot \left(\tilde{c}^2 - \tilde{a}^2\, tg^2\,\bar{\varphi}_{ij}\right)$$

During calculations it is useful to adopt auxiliary marking

$$F_{ij} = \bar{z}_i - tg\,\bar{\varphi}_{ij}\left(\bar{x}_i\, cos\,\bar{\alpha}_{ij} + \bar{y}_{ij}\, sin\,\bar{\alpha}_{ij}\right)$$
$$G_{ij} = tg^2\,\bar{\varphi}_{ij}$$
$$H_{ij} = \left(\bar{x}_i\, cos\,\bar{\alpha}_{ij} - \bar{y}_{ij}\, sin\,\bar{\alpha}_{ij}\right)^2$$

$$\frac{\partial\Phi_{ij}}{\partial a} = 2\tilde{a}\left(z_0 - F_{ij}\right)^2 + \left(\tilde{c}^2 - \tilde{a}^2 G_{ij}\right) - G_{ij}\cdot\left(\tilde{a}^2 - H_{ij}\right)$$

$$\frac{\partial\Phi_{ij}}{\partial c} = 2\tilde{c}\left[\tilde{a}^2 - H_{ij}\right]$$

$$\frac{\partial\Phi_{ij}}{\partial z_0} = 2\tilde{a}^2\left[\tilde{z}_0 - F_{ij}\right]$$

$$\Phi_{ij0} = \tilde{a}^2\left[\tilde{z}_0 - F_{ij}\right]^2 + \left[\tilde{a}^2 - H_{ij}\right]\cdot\left(\tilde{c}^2 - \tilde{a}^2 G_{ij}\right)$$

Finally, a system of equations Ax = L will be built, where

$$\mathbf{A} = \begin{bmatrix} \frac{\partial\Phi_{11}}{\partial a} & \frac{\partial\Phi_{11}}{\partial c} & \frac{\partial\Phi_{11}}{\partial z_0} \\ \frac{\partial\Phi_{12}}{\partial a} & \frac{\partial\Phi_{12}}{\partial c} & \frac{\partial\Phi_{12}}{\partial z_0} \\ .. & ... & ... \\ \frac{\partial\Phi_{sr_i}}{\partial a} & \frac{\partial\Phi_{sr_i}}{\partial c} & \frac{\partial\Phi_{sr_i}}{\partial z_0} \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} d\xi \\ d\eta \\ d\zeta \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} -\Phi_{110} \\ -\Phi_{120} \\ \dots \\ -\Phi_{sr0} \end{bmatrix}$$

In result of a solution of this system one will get estimators of parameters with a full assessment of accuracy.

Deviations from the model surface can be determined according to the formulas

$$d_{ij} = \Delta\alpha_{ij} l_{ij}$$

$$d_{ij} = \frac{\delta_{ij}}{2\sqrt{H_{ij}}\left(c^2 - a^2 G_{ij}\right)}$$

CONCLUSION

The problem with determination of the shape and situation of second degree surfaces stall exists. This study is devoted to the conical intersection method which, although little known, offers interesting solutions. Conical intersections can be characterised by the fact that it is easy to measure them and to program calculations, it is easy to measure the accuracy of calculated parameters and the fact that there are many stages allows to control intermediate results. The algorithms presented here have features of universality. They could be used both to analyse shapes of shell objects and, for example, to determine deviations of such objects as industrial chimneys, masts and piles.

The algorithms that are presented in the study have been verified, checked on the basis of test objects and on results from real measurements of chimney cooler situated in 'Łaziska' power plant.

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