ESTIMATION LINEAR MODEL USING BLOCK GENERALIZED INVERSE OF A MATRIX

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ABSTRACT

The work shows the principle of generalized linear model, point estimation, which can be used as a basis for determining the status of movements and deformations of engineering objects. The structural model can be put on any boundary conditions, for example, to ensure the continuity of the deformations. Estimation by the method of least squares was carried out taking into account the terms and conditions of the Gauss-Markov for quadratic forms stored using Lagrange function. The original solution is removal of the generalized inverse of block matrix according to the designs of the fourth degree.

Keywords: generalized linear model, point estimate, block matrixes, deformation, displacement.

INTRODUCTION

Geodetic issues often have a problem with understanding the results of measurements, which are functional associated with each other and with parameters describing the studied phenomenon [1], [4]. If the observation vector is denoted by L_i, random deviations by δ_i and parameters by Θ_j, then the function compounds between these vectors can be written as:

$$x(L_i - \delta_i, \Theta_j) = y(i = 1,2,\ldots,n),(j = 1,2,\ldots,n) \tag{1}$$

The separation of variables on the nature of the random variables that are parameters of the model is very important. Appropriate choice depends on how the data acquisition and the nature of the phenomena concerned. If between the selected variables will determine the function conditions, but the values of variables can be specified only by a method of successive approximations, you will want to consider whether these variables should not be treated as parameters of the model. Similarly, if the variables occurring in relationships have, on the basis of the assigned function, the values of a small precision, whether to treat them as random variables? [2].

A variable of a random can be assigned two values: observed and model. For the parameters, especially in engineering, additional restrictions may be imposed. The relationship between these values is determined by the random displacement δ, that is $L = \hat{L} + \delta \Rightarrow E(L) = \hat{L}$. From this it follows that the function key (1) in terms of random variables is the value model.

GENERALIZED LINEAR MODEL

Any system of linear equations can be written in matrix form [6]

$$\mathbf{F}\,\mathbf{z} = \mathbf{g} \tag{2}$$

where:

- **F** – the coefficients matrix representing the value of the first derivative for the approximate values of the estimated parameters of the model, with dimensions $(m \times n + u)$;
- **z** – the array of unknowns, representing the random deviation or increases to the approximate values of the estimated parameters of the model, with dimensions $(n + u \times 1)$;
- **g** – the residues vector, resulting from the observed values of variables random and approximate values of the estimated parameters of the model, with dimensions $(m \times 1)$.

Equation (2) can be solved for a variety of conditions arising from the specific boundary phenomenon [3], [7]. Boundary conditions will lead to a vertical or a horizontal division of equation (2), and to systems of equations.
Vertical partition (2) arises from the imposition on the part of unknown condition – while Markov Gauss treat the rest of the unknowns as parameters of the model. Such equations can be written in the form of a block:

$$[\mathbf{CA}]\begin{bmatrix}\mathbf{v}\\ \mathbf{x}\end{bmatrix} = \mathbf{u} \tag{3}$$

What is equivalent to an equation

$$\mathbf{Cv} + \mathbf{Ax} = \mathbf{u} \tag{4}$$

while
v - vector of random deviations,
x - vector of parameters of the model.

For horizontal distribution of equations (2) we will use the form (3), in which the model parameter values impose restrictions $\mathbf{Bx} = \mathbf{w}$. Having regard to this condition, the system of equations (3) can be written in the form of block:

$$\begin{bmatrix}\mathbf{C} & \mathbf{A}\\ \mathbf{0} & \mathbf{B}\end{bmatrix}\begin{bmatrix}\mathbf{v}\\ \mathbf{x}\end{bmatrix} = \begin{bmatrix}\mathbf{u}\\ \mathbf{w}\end{bmatrix} \tag{5}$$

Horizontal Division equations (2), (5), may result boundary condition in which the upper part of the equations must meet probabilistic model, and the lower part of the equations should comply with the deterministic.

Determination of matrix in equation (5):

- $\underset{(w,\,n)}{\mathbf{C}}$ – the matrix coefficients of the vector of random deviations **v**;
- $\underset{(w,\,1)}{\mathbf{v}}$ – the vector of random deviations for probability model;
- $\underset{(w,\,p)}{\mathbf{A}}$ – the matrix coefficients by estimated vector of model parameters (**x**) in a probability part;
- $\underset{(p,\,1)}{\mathbf{x}}$ – the vector of unknowns, i.e., the estimated parameters (increases to the approximate values of the estimated parameters), occurring in the probabilistic model and the deterministic model;

$\underset{(w,1)}{\mathbf{u}}$ – residual vector in parts of a probability model;

$\underset{(r,p)}{\mathbf{B}}$ – the matrix coefficients of the vector x in the deterministic model parameter deterministic;

$\underset{(r,1)}{\mathbf{w}}$ – residual vector in the model deterministic.

In accordance with the principle of the method of least squares unencumbered estimators system equations (5) have to fulfill Gauss–Markov conditions arising from the square forms stored using Lagrange

$$\Psi^2 = \min_{\hat{\mathbf{w}} \cap \mathbf{Bx}=\mathbf{w}} \left\{ \left(\mathbf{t}-\hat{\mathbf{t}}\right)^{\mathrm{T}} \mathbf{Cov}(\mathbf{t})^{-1} \left(\mathbf{t}-\hat{\mathbf{t}}\right) + 2\mathbf{k}^{\mathrm{T}} \left[\mathbf{C}\left(\mathbf{t}-\hat{\mathbf{t}}\right) + \mathbf{Ax} - \mathbf{u} \right] \right\} \tag{6}$$

Quadratic form (6) can be written in as

$$\Psi^2 = \mathbf{v}^{\mathrm{T}}\mathbf{Pv} + 2\mathbf{k}^{\mathrm{T}}\left(\mathbf{Cv} + \mathbf{Ax} - \mathbf{u}\right) + 2\mathbf{j}^{\mathrm{T}}\left(\mathbf{Bx} - \mathbf{w}\right) = \min \tag{7}$$

In these forms, there are the following designations:

t – random variable vector, which is the size of the observed $(n \times 1)$;

$\hat{\mathbf{t}}$ – Vector of model value of vector **t** $(n \times 1)$ about expected value $\mathrm{E}(\hat{\mathbf{w}}) = \mathbf{w}$;

v – random probabilistic model for vector $(n \times 1)$ associated with explanatory variable, that is, from observations **w** and their exemplary values $\hat{\mathbf{t}}$ the equation for the $\mathbf{v} = \mathbf{t} - \hat{\mathbf{t}}$, about expected value $\mathrm{E}(\mathbf{v}) = 0$;

$\mathbf{Cov}(\mathbf{t})$ – covariance matrix for the values observed $(n \times n)$;

P – weight matrix with determinant factor for different from zero $(n \times n)$, corresponding to the inverse of the covariance matrix, which is $\mathbf{P} = \mathbf{Cov}(\mathbf{t})^{-1}$;

j, k – Lagrange coefficient vectors;

C – the matrix coefficients of the vector a random variable, the resulting from the conditions imposed on the observation vector **t**;

x – the estimated model parameters vector, in part of a probability and deterministic $(u \times 1)$;

A – the matrix coefficients $(m \times u)$ by estimated vector parameters **x** (in probability), resulting from the binding function observation vector **t** with model parameters **x** taking into account the **C** matrix;

u – the value defining the function conditions $\mathbf{C}\left(\mathbf{t}-\hat{\mathbf{t}}\right) + \mathbf{Ax}$ imposed on the vector **t**;

B – the matrix of coefficients of the function conditions imposed only on the parameters **x** (that is, resulting from the nature of the phenomenon, such as deformation compatibility conditions);

w – The rest vector in deterministic part $(r \times 1)$.

The value of the observed should be linked to estimated parameters functional conditions, which can be written in the form in general: $\mathbf{C}(\mathbf{t}-\hat{\mathbf{t}})+\mathbf{A}\mathbf{x}=\mathbf{u}$.

Model (7) may include not only observed the value **t** and estimated data parameters **x**, but, for example, parameters changing in time (speed). Hence the additional conditions should be met, for example the conditions of geometric figures, in case of a geodetic network model analysis. Let's show those conditions by using a vector of random deviations **v**, which represents increases to the value of the model size of the watch, in other words $\mathbf{v}=\mathbf{t}-\hat{\mathbf{t}}$.

After applying the above view, the form of square (6) is minimised to a random vector **v** and the function terms with do not include a random component.

$$\mathbf{B}\mathbf{x}=\mathbf{w} \tag{8}$$

Matrix **C** can arise from the conditions imposed by the figures on the value of the random variable vector model **t**. Parameters vector **x** can describe a vector of coefficients field movements. It's worth pointing out that approximate values of parameters **x**, describing investigated this phenomenon, are not generally well known, hence the functions (7) with respect to these parameters should be in the form of liner.

The matrix **A** is determined by the coefficients estimated the parameters **x**, but taking into account the additional conditions, for example the conditions of figure, which is included in matrix **C**. If matrix function expressions occurring in (7), but containing only variables x_1, x_2, x_3 i t, we shall arrange in the form of a matrix $\mathbf{A}_0$, then $\mathbf{A}=\mathbf{C}\mathbf{A}_\mathbf{0}$.

Functional conditions (8) in respect of which follows the minimization of quadratic forms (6), due to the nature of the phenomenon. In the analysis of deformations of object-matching conditions of deformation, i.e. with fixed binding components of deformation tensor.

POINT ESTIMATION MODEL OF GENERALIZED

A prerequisite for the existence of a minimum of Lagrange function (7) is to reset the first vector derivatives, but sufficient one (subject to necessary) is to positively referred to matrices resulting from the second derivatives [5]. The necessary conditions shall therefore:

$$\frac{\partial\Psi}{\partial\mathbf{v}}=2\mathbf{P}\mathbf{v}+2\mathbf{C}^{\mathrm{T}}\mathbf{k}=\mathbf{0}$$

$$\frac{\partial\Psi}{\partial\mathbf{x}}=2\mathbf{A}^{\mathrm{T}}\mathbf{k}+2\mathbf{B}^{\mathrm{T}}\mathbf{j}=\mathbf{0}$$

$$\frac{\partial\Psi}{\partial\mathbf{k}}=2\left(\mathbf{C}\mathbf{v}+\mathbf{A}\mathbf{x}-\mathbf{u}\right)=\mathbf{0}$$

$$\frac{\partial\Psi}{\partial\mathbf{j}}=2\left(\mathbf{B}\mathbf{x}-\mathbf{w}\right)=\mathbf{0}$$

which leads to the four matrix equations:

$$\begin{aligned}&\mathbf{P}\mathbf{v}+\mathbf{C}^{\mathrm{T}}\mathbf{k}=\mathbf{0}\\&\mathbf{A}^{\mathrm{T}}\mathbf{k}+\mathbf{B}^{\mathrm{T}}\mathbf{j}=\mathbf{0}\\&\mathbf{C}\mathbf{v}+\mathbf{A}\mathbf{x}=\mathbf{u}\\&\mathbf{B}\mathbf{x}=\mathbf{w}\end{aligned} \tag{9}$$

Equations (9) written by using block-matrix takes the form

$$\begin{bmatrix}\dfrac{\partial\Psi}{\partial\mathbf{v}}\\\dfrac{\partial\Psi}{\partial\mathbf{k}}\\\dfrac{\partial\Psi}{\mathbf{x}}\\\dfrac{\partial\Psi}{\mathbf{j}}\end{bmatrix}=\begin{bmatrix}\mathbf{P}&\mathbf{C}^{\mathrm{T}}&\mathbf{0}&\mathbf{0}\\\mathbf{C}&\mathbf{0}&\mathbf{A}&\mathbf{0}\\\mathbf{0}&\mathbf{A}^{\mathrm{T}}&\mathbf{0}&\mathbf{B}^{\mathrm{T}}\\\mathbf{0}&\mathbf{0}&\mathbf{B}&\mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{v}\\\mathbf{k}\\\mathbf{x}\\\mathbf{j}\end{bmatrix}-\begin{bmatrix}\mathbf{0}\\\mathbf{u}\\\mathbf{0}\\\mathbf{w}\end{bmatrix}=\begin{bmatrix}\mathbf{0}\\\mathbf{0}\\\mathbf{0}\\\mathbf{0}\end{bmatrix} \tag{10}$$

Sufficient conditions for the existence of a minimum of proper are met, because the matrices $\mathbf{P},\mathbf{C}\mathbf{C}^{\mathrm{T}},\mathbf{P}\mathbf{A}^{\mathrm{T}}\mathbf{A},\mathbf{C}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}$ resulting from the second partial derivatives are positively identified, provided that they are not peculiar. If any of these matrix determinant is equal zero, there are infinitely many solutions. By imposing additional conditions on the estimated data parameters shall be unambiguous solution (minimum).

The solution of the system (10) will be brought to determinate the generalized inverse of the main block matrix made by $4\times4=16$ sub-matrixes. Because the main block matrix is symmetric, it is also her generalized inverse of block matrix will be symmetric, i.e.

$$\begin{bmatrix}\mathbf{P}&\mathbf{C}^{\mathrm{T}}&\mathbf{0}&\mathbf{0}\\\mathbf{C}&\mathbf{0}&\mathbf{A}&\mathbf{0}\\\mathbf{0}&\mathbf{A}^{\mathrm{T}}&\mathbf{0}&\mathbf{B}^{\mathrm{T}}\\\mathbf{0}&\mathbf{0}&\mathbf{B}&\mathbf{0}\end{bmatrix}^{-}=\begin{bmatrix}\mathbf{Q}_1&\mathbf{Q}_2&\mathbf{Q}_3&\mathbf{Q}_4\\\mathbf{Q}_2^{\mathrm{T}}&\mathbf{Q}_5&\mathbf{Q}_6&\mathbf{Q}_7\\\mathbf{Q}_3^{\mathrm{T}}&\mathbf{Q}_6^{\mathrm{T}}&\mathbf{Q}_8&\mathbf{Q}_9\\\mathbf{Q}_4^{\mathrm{T}}&\mathbf{Q}_7^{\mathrm{T}}&\mathbf{Q}_9^{\mathrm{T}}&\mathbf{Q}_{10}\end{bmatrix}$$

which is equal to what is written below:

$$\begin{bmatrix}\mathbf{P}&\mathbf{C}^{\mathrm{T}}&\mathbf{0}&\mathbf{0}\\\mathbf{C}&\mathbf{0}&\mathbf{A}&\mathbf{0}\\\mathbf{0}&\mathbf{A}^{\mathrm{T}}&\mathbf{0}&\mathbf{B}^{\mathrm{T}}\\\mathbf{0}&\mathbf{0}&\mathbf{B}&\mathbf{0}\end{bmatrix}\times\begin{bmatrix}\mathbf{Q}_1&\mathbf{Q}_2&\mathbf{Q}_3&\mathbf{Q}_4\\\mathbf{Q}_2^{\mathrm{T}}&\mathbf{Q}_5&\mathbf{Q}_6&\mathbf{Q}_7\\\mathbf{Q}_3^{\mathrm{T}}&\mathbf{Q}_6^{\mathrm{T}}&\mathbf{Q}_8&\mathbf{Q}_9\\\mathbf{Q}_4^{\mathrm{T}}&\mathbf{Q}_7^{\mathrm{T}}&\mathbf{Q}_9^{\mathrm{T}}&\mathbf{Q}_{10}\end{bmatrix}=\begin{bmatrix}\mathbf{I}_{\mathrm{h}}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{I}_{\mathrm{h}}&\mathbf{0}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{I}_{\mathrm{h}}&\mathbf{0}\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{I}_{\mathrm{h}}\end{bmatrix} \quad 11)$$

where $\mathbf{I}_{\mathrm{h}}$ the unit is an Hertmitian matrix.

The degree of the individual matrix $\underset{(n+w+p+r,\,n+w+p+r)}{\mathbf{Q}}$ according to equations (10).

On the basis of the agreement (11) we obtain sixteen equations about ten unknowns

$$\mathbf{P}\ \mathbf{Q}_1+\mathbf{C}^{\mathrm{T}}\mathbf{Q}_2^{\mathrm{T}}=\mathbf{I}_{\mathrm{h}} \tag{12}$$

$$\mathbf{P}\ \mathbf{Q}_2 + \mathbf{C}^{\mathrm{T}}\mathbf{Q}_5 = \mathbf{0} \tag{13}$$
$$\mathbf{P}\ \mathbf{Q}_3 + \mathbf{C}^{\mathrm{T}}\mathbf{Q}_6 = \mathbf{0} \tag{14}$$
$$\mathbf{P}\ \mathbf{Q}_4 + \mathbf{C}^{\mathrm{T}}\mathbf{Q}_7 = \mathbf{0} \tag{15}$$
$$\mathbf{C}\ \mathbf{Q}_1 + \mathbf{A}\ \mathbf{Q}_3^{\mathrm{T}} = \mathbf{0} \tag{16}$$
$$\mathbf{C}\ \mathbf{Q}_2 + \mathbf{A}\ \mathbf{Q}_6^{\mathrm{T}} = \mathbf{I}_{\mathrm{h}} \tag{17}$$
$$\mathbf{C}\ \mathbf{Q}_3 + \mathbf{A}\ \mathbf{Q}_8 = \mathbf{0} \tag{18}$$
$$\mathbf{A}^{\mathrm{T}}\mathbf{Q}_2^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}\mathbf{Q}_4^{\mathrm{T}} = \mathbf{0} \tag{19}$$
$$\mathbf{C}\ \mathbf{Q}_4 + \mathbf{A}\ \mathbf{Q}_9 = \mathbf{0} \tag{20}$$
$$\mathbf{A}^{\mathrm{T}}\mathbf{Q}_5 + \mathbf{B}^{\mathrm{T}}\mathbf{Q}_7^{\mathrm{T}} = \mathbf{0} \tag{21}$$
$$\mathbf{A}^{\mathrm{T}}\mathbf{Q}_6 + \mathbf{B}^{\mathrm{T}}\mathbf{Q}_9^{\mathrm{T}} = \mathbf{I}_{\mathrm{h}} \tag{22}$$
$$\mathbf{A}^{\mathrm{T}}\mathbf{Q}_7 + \mathbf{B}^{\mathrm{T}}\mathbf{Q}_{10} = \mathbf{0} \tag{23}$$
$$\mathbf{B}\ \mathbf{Q}_3^{\mathrm{T}} = \mathbf{0} \tag{24}$$
$$\mathbf{B}\ \mathbf{Q}_6^{\mathrm{T}} = \mathbf{0} \tag{25}$$
$$\mathbf{B}\ \mathbf{Q}_8 = \mathbf{0} \tag{26}$$
$$\mathbf{B}\ \mathbf{Q}_9 = \mathbf{I}_{\mathrm{h}} \tag{27}$$

At first they were matrices $\mathbf{Q}_4, \mathbf{Q}_7, \mathbf{Q}_9, \mathbf{Q}_{10}$, on the basis of the equations (15), (20), (23) i (27). Comments Were Received:

$$\mathbf{Q}_4 = -\mathbf{P}^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{Q}_7,$$
$$\mathbf{Q}_7 = \left(\mathbf{C}\ \mathbf{P}^{-1}\mathbf{C}^{\mathrm{T}}\right)^{-}\mathbf{A}\ \mathbf{Q}_9,$$
$$\mathbf{Q}_9 = -\left(\mathbf{A}^{\mathrm{T}}\left(\mathbf{C}\ \mathbf{P}^{-1}\mathbf{C}^{\mathrm{T}}\right)^{-}\mathbf{A}\right)^{-}\mathbf{B}^{\mathrm{T}}\mathbf{Q}_{10},$$
$$\mathbf{Q}_{10} = -\left(\mathbf{B}\left(\mathbf{A}^{\mathrm{T}}\left(\mathbf{C}\ \mathbf{P}^{-1}\mathbf{C}^{\mathrm{T}}\right)^{-}\mathbf{A}\right)^{-}\mathbf{B}^{\mathrm{T}}\right)^{-}.$$

Based on equation (14), (18) and (22) they were matrices:

$$\mathbf{Q}_{3.} = -\mathbf{P}^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{Q}_6,$$
$$\mathbf{Q}_6 = \left(\mathbf{C}\mathbf{P}^{-1}\mathbf{C}^{\mathrm{T}}\right)^{-}\mathbf{A}\ \mathbf{Q}_8,$$
$$\mathbf{Q}_8 = \left(\mathbf{A}^{\mathrm{T}}\left(\mathbf{C}\mathbf{P}^{-1}\mathbf{C}^{\mathrm{T}}\right)^{-}\mathbf{A}\right)^{-}\left(\mathbf{I}_{\mathrm{h}} - \mathbf{B}^{\mathrm{T}}\mathbf{Q}_9^{\mathrm{T}}\right).$$

Another matrix equations (13) and (17) allows the calculation of:

$$\mathbf{Q}_2 = -\mathbf{P}^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{Q}_5,$$
$$\mathbf{Q}_5 = -\left(\mathbf{C}\ \mathbf{P}^{-1}\mathbf{C}^{\mathrm{T}}\right)^{-}\left(\mathbf{I} - \mathbf{A}\ \mathbf{Q}_6^{\mathrm{T}}\right).$$

Matrix $\mathbf{Q}_1$ was established on the basis of the equation (12):

$$\mathbf{Q}_1 = \mathbf{P}^{-1}\left(\mathbf{I}_{\mathrm{h}} - \mathbf{C}^{\mathrm{T}}\mathbf{Q}_2^{\mathrm{T}}\right).$$

After the introduction of the determinations:

$$\underset{(w,\,w)}{\mathbf{N}} = \underset{(w,\,n)}{\mathbf{C}}\,\underset{(n,\,n)}{\mathbf{P}^{-1}}\,\underset{(n,\,w)}{\mathbf{C}^{\mathrm{T}}}$$

$$\underset{(p,\,p)}{\mathbf{M}} = \underset{(p,\,w)}{\mathbf{A}^{\mathrm{T}}}\,\underset{(w,\,w)}{\mathbf{N}^{-}}\,\underset{(w,\,p)}{\mathbf{A}}$$

We get:

$$\underset{(r,\,r)}{\mathbf{Q}_{10}} = -\left(\underset{(r,\,p)}{\mathbf{B}}\,\underset{(p,\,p)}{\mathbf{M}^{-}}\,\underset{(p,\,r)}{\mathbf{B}^{\mathrm{T}}}\right)^{-}$$

$$\underset{(p,\,r)}{\mathbf{Q}_{9}} = -\underset{(p,\,p)}{\mathbf{M}^{-}}\,\underset{(p,\,r)}{\mathbf{B}^{\mathrm{T}}}\,\underset{(r,\,r)}{\mathbf{Q}_{10}}$$

$$\underset{(p,\,p)}{\mathbf{Q}_{8}} = \underset{(p,\,p)}{\mathbf{M}^{-}}\left(\underset{(p,\,p)}{\mathbf{I}_{\mathrm{h}}} - \underset{(p,\,r)}{\mathbf{B}^{\mathrm{T}}}\,\underset{(r,\,p)}{\mathbf{Q}_{9}^{\mathrm{T}}}\right)$$

$$\underset{(w,\,r)}{\mathbf{Q}_{7}} = \underset{(w,\,w)}{\mathbf{N}^{-}}\,\underset{(w,\,p)}{\mathbf{A}}\,\underset{(p,\,p)}{\mathbf{Q}_{9}}$$

$$\underset{(w,\,p)}{\mathbf{Q}_{6}} = \underset{(w,\,w)}{\mathbf{N}^{-}}\,\underset{(w,\,p)}{\mathbf{A}}\,\underset{(p,\,p)}{\mathbf{Q}_{8}}$$

$$\underset{(w,\,w)}{\mathbf{Q}_{5}} = -\underset{(w,\,w)}{\mathbf{N}^{-}}\left(\underset{(w,\,w)}{\mathbf{I}_{\mathrm{h}}} - \underset{(w,\,p)}{\mathbf{A}}\,\underset{(p,\,w)}{\mathbf{Q}_{6}^{\mathrm{T}}}\right)$$

$$\underset{(n,\,r)}{\mathbf{Q}_{4}} = -\underset{(n,\,n)}{\mathbf{P}^{-1}}\,\underset{(n,\,w)}{\mathbf{C}^{\mathrm{T}}}\,\underset{(w,\,p)}{\mathbf{Q}_{7}}$$

$$\underset{(n,\,p)}{\mathbf{Q}_{3}} = -\underset{(n,\,n)}{\mathbf{P}^{-1}}\,\underset{(n,\,w)}{\mathbf{C}^{\mathrm{T}}}\,\underset{(w,\,p)}{\mathbf{Q}_{6}}$$

$$\underset{(n,\,w)}{\mathbf{Q}_{2}} = -\underset{(n,\,n)}{\mathbf{P}^{-1}}\,\underset{(n,\,w)}{\mathbf{C}^{\mathrm{T}}}\,\underset{(w,\,w)}{\mathbf{Q}_{5}}$$

$$\underset{(n,\,n)}{\mathbf{Q}_{1}} = \underset{(n,\,n)}{\mathbf{P}^{-1}}\left(\underset{(n,\,n)}{\mathbf{I}_{\mathrm{h}}} - \underset{(n,\,w)}{\mathbf{C}^{\mathrm{T}}}\,\underset{(w,\,n)}{\mathbf{Q}_{2}^{\mathrm{T}}}\right)$$

Above sub matrixes $\mathbf{Q}_{\mathrm{i}}$ satisfy the equations (12-27).

Seeking solutions matrix of the system (10) can be expressed as the product of the following block matrix

$$\begin{bmatrix} \hat{\mathbf{v}} \\ \hat{\mathbf{k}} \\ \hat{\mathbf{x}} \\ \hat{\mathbf{j}} \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 & \mathbf{Q}_3 & \mathbf{Q}_4 \\ \mathbf{Q}_2^{\mathrm{T}} & \mathbf{Q}_5 & \mathbf{Q}_6 & \mathbf{Q}_7 \\ \mathbf{Q}_3^{\mathrm{T}} & \mathbf{Q}_6^{\mathrm{T}} & \mathbf{Q}_8 & \mathbf{Q}_9 \\ \mathbf{Q}_4^{\mathrm{T}} & \mathbf{Q}_7^{\mathrm{T}} & \mathbf{Q}_9^{\mathrm{T}} & \mathbf{Q}_{10} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \mathbf{u} \\ \mathbf{0} \\ \mathbf{w} \end{bmatrix}$$

Free estimates given the uncertainties of the general model are:

$$\hat{\mathbf{v}} = \mathbf{Q_2 u} + \mathbf{Q_4 w}$$

$$\hat{\mathbf{x}} = \mathbf{Q_6^T u} + \mathbf{Q_9 w}$$

Vectors of coefficients can be calculated according to the Lagrange:

$$\hat{\mathbf{k}} = \mathbf{Q_5 u} + \mathbf{Q_7 w}$$

$$\hat{\mathbf{j}} = \mathbf{Q_7^T u} + \mathbf{Q_{10} w}$$

An estimate of the size of the model $\hat{\mathrm{t}}$ is designated according to the formula:

$$\hat{\mathbf{t}} = \mathbf{w} - \hat{\mathbf{v}} = \mathbf{w} - \left(\mathbf{Q}_2\mathbf{u} + \mathbf{Q}_4\mathbf{w}\right)$$

Unladed variance estimator is given by equation

$$\hat{\sigma}^2 = \frac{\mathbf{v}^{\mathrm{T}}\mathbf{P}\,\mathbf{v}}{\mathrm{R}(\mathbf{C}\mathbf{P}^{-1}\mathbf{C}^{\mathrm{T}}) - \mathrm{R}(\mathbf{A}\,|\,\mathbf{B})}$$

Covariance matrices, vectors $\hat{\mathbf{x}}, \hat{\mathbf{v}}$ express the following dependencies:

$$\mathbf{Cov}(\hat{\mathbf{v}}) = \hat{\sigma}^2\mathbf{Q_1}$$
$$\mathbf{Cov}(\hat{\mathbf{x}}) = \hat{\sigma}^2\mathbf{Q_8}$$

Estimators $\hat{\mathbf{v}}$ *i* $\hat{\mathbf{x}}$ can be written in the form:

$$\hat{\mathbf{v}} = \mathbf{Q}_2\mathbf{C}\mathbf{v} + \mathbf{Q}_2\mathbf{A}\mathbf{x} + \mathbf{Q}_4\mathbf{B}\mathbf{x} = \mathbf{v}^{\mathrm{o}} + \Delta\mathbf{v}_{\mathrm{a}} + \Delta\mathbf{v}_{\mathrm{b}}$$
$$\hat{\mathbf{x}} = \mathbf{Q}_6^{\mathrm{T}}\mathrm{C}\mathbf{v} + \mathbf{Q}_6^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{Q}_9\mathbf{B}\mathbf{x} = \mathbf{x}^{\mathrm{o}} + \Delta\mathbf{x}_{\mathrm{a}} + \Delta\mathbf{x}_{\mathrm{b}}$$

Vectors $\Delta\mathbf{v}_{\mathrm{a}}$ i $\Delta\mathbf{x}_{\mathrm{a}}$ resulting from the adoption of unknowns for the model parameters, vectors $\Delta\mathbf{v}_{\mathrm{b}}$ i $\Delta\mathbf{x}_{\mathrm{b}}$ due to the restrictions imposed on the parameters of the model. If all the unknowns may enforce Gauss Markov-condition, then $\Delta\mathbf{v}_{\mathrm{a}} = 0$ i $\Delta\mathbf{x}_{\mathrm{a}} = 0$. If not, then the boundary conditions are $\Delta\mathbf{v}_{\mathrm{b}} = 0$ i $\Delta\mathbf{x}_{\mathrm{b}} = 0$.

On the basis of the presented solutions to equations (9) can be used to analyze specific cases of linear models, which arise from the observation conditions and the nature of the phenomenon.

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