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## IS THE GENERALIZED PADÉ APPROXIMATION OF THE DEAD TIME APPLICABLE TO STABILITY MARGINS ANALYSIS?

**Keywords:** Stability analysis, fractional-order systems, time-delay, Padé approximation

### 1. INTRODUCTION

Presence of a dead time in any of the control loops has been a subject of research for many years, and resulted in integer-order rational function approximation of the time delay, named Padé approximation [7]. The dead time is given rise to by many reasons, e.g., transport delay or remote measurements, resulting in the response of the control system to be dependent on its previous values. In control domain, it may cause either performance deterioration or even instability of the control system, what makes it undesirable, and, yet, ubiquitous (many engineering plants and processes cannot be described accurately without the introduction of delay elements).

As stated in [11], systems with dead time have been studied using discretization techniques with an extended state vector, as in [9]. A study on stability and performance of systems with dead time is given in [1], and is performed by considering the roots of the closed-loop characteristic equation. One can also find a survey of stability analysis methods presented in [6]. From [7], it is difficult to analyze stability and design controllers for systems with dead times, since it is impossible to employ transfer function-based analysis of the closed-loop system.

One of the most common methods to overcome the latter, is to introduce an integer-order Padé approximation, to make the transfer function of the time-delay operator rational again, and to analyze the obtained system by means of standard methods. This approximation provides a finite-dimensional rational approximation of the dead time and has been applied in various situations [3, 7, 10]. This method is also reliable, since as far as stability analysis is concerned, it presents results very close to the true ones.

As remarked in [14], there already are some results available on fractional-order approximation of the time delay, and the only problem is that Padé approximation cannot be the representative of a fractional-order dead-time operator in a standardized form. However, the generalized Padé approximation using the fractional-order transfer function is capable of obtaining better approximate performance, though in time domain. This paper aims to verify, whether the following statement can be extended to stability analysis.

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The paper [14] presents the fractional-order Padé approximation of the time-delay operator, and this paper aims to analyze the applicability of the method to analysis of stability of the closed-loop systems with dead times in feedforward loop.

A great deal of publications has been dedicated to the issue of stability of fractional-order systems, just to mention [5, 8, 12] or [4]. This time, however, the stability considerations must refer to the integer-order time-delayed system.

The paper is organized as follows: Section II presents basic notation, Section III cites the main results of [14] and [7], Section IV includes analysis of the methods, with conclusions given in Section V.

## 2. BASIC NOTATION, STATEMENT OF THE PROBLEM

The Riemann-Liouville fractional-order integral of order  $a$  of a function  $f(t)$  is defined by [14]

$$\mathcal{I}_a f(t) = \frac{1}{\Gamma(a)} \int_0^t (t - \tau)^{a-1} f(\tau) d\tau, \quad (1)$$

where the Gamma (factorial) function is defined as

$$\Gamma(a) = \int_0^\infty y^{a-1} e^{-y} dy. \quad (2)$$

The Caputo fractional-order derivative of order  $a$ , where  $n - 1 \leq a < n$  and  $n \in \mathcal{Z}_+$  of a function  $f(t)$ , is defined by

$$\mathcal{D}^a f(t) = \frac{1}{\Gamma(n - a)} \int_a^t (t - \tau)^{n-a-1} f^{(n)}(\tau) d\tau. \quad (3)$$

These operators are used by FOMCON toolbox, and are approximated by their discrete-time counterparts, to enable Matlab to conduct calculations, when performing simulations of a fractional-order system.

The standard approximation of the transport lag (dead time) by Henri Padé matches the expansion of  $e^{-sT_0}$  function to the expansion of a rational function of orders  $(p, q)$ . First, let us approximate  $e^{-s}$ , and then let us introduce the approximation of  $e^{-sT_0}$ , where  $T_0$  [sec] is a delay time. To find the  $(1, 1)$  approximation, one needs to find the coefficients  $b_0, b_1, a_1$  that result in small error defined as [3, 7]

$$\varepsilon = e^{-s} - \frac{b_1 s + b_0}{a_1 s + 1}. \quad (4)$$

Expanding the exponential function into McLaurin series:

$$e^{-s} = 1 - s + \frac{s^2}{2!} - \frac{s^3}{3!} + \frac{s^4}{4!} - \dots,$$

$$\frac{b_1 s + b_0}{a_1 s + 1} = b_0 + (b_1 - a_1 b_0)s - a_1(b_1 - a_1 b_0)^2 + a_1^2(b_1 - a_1 b_0)s^3 + \dots,$$

and introducing the following equations:

$$\begin{aligned} b_0 &= 1, \\ b_1 - a_1 b_0 &= -1, \\ -a_1(b_1 - a_1 b_0) &= \frac{1}{2}, \\ a_1^2(b_1 - a_1 b_0) &= -\frac{1}{6}, \end{aligned}$$

by matching the first 3 elements of the sum and substituting  $sT_0$  instead of  $s$ , the Padé approximation (1, 1) becomes:

$$e^{-sT_0} \approx \frac{1 - s \frac{T_0}{2}}{1 + s \frac{T_0}{2}}. \quad (5)$$

The (2, 2) approximation (with 5 parameters) becomes:

$$e^{-sT_0} \approx \frac{1 - s \frac{T_0}{2} + s^2 \frac{T_0^2}{12}}{1 + s \frac{T_0}{2} + s^2 \frac{T_0^2}{12}}, \quad (6)$$

and etc.

In general, the standard Padé  $(p, q)$  approximation can be expressed as

$$G_P(s) = \frac{b_0 + b_1(sT_0) + b_2(sT_0)^2 + \dots + b_q(sT_0)^q}{a_0 + a_1(sT_0) + a_2(sT_0)^2 + \dots + a_p(sT_0)^p} \quad (7)$$

with  $q \leq p$ , and

$$b_i = (-1)^i \frac{(q+p-i)!q!}{(q+p)!(q-i)!i!} \quad (i = 0, 1, 2, \dots, q), \quad (8)$$

$$a_j = \frac{(q+p-j)!p!}{(q+p)!(q-j)!j!} \quad (i = 0, 1, 2, \dots, p). \quad (9)$$

When  $p = q$ , the poles and zeros of  $G_P(s)$  are symmetrical w.r.t. the imaginary axis, and the coefficients of the symmetrical Padé approximation are given by [14]

$$b_i = (-1)^i p_i \quad (i = 0, 1, 2, \dots, p). \quad (10)$$

The approximation is usually applied whenever there is a need to calculate approximate locations of closed-loop poles, and when any of the loops contains dead time.

Now, let the closed-loop system with the structure as in Figure 1 be considered.

The closed-loop transfer function of the system presented in Figure 1 is given by

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}, \quad (11)$$

and, in general, either  $G(s)$  or  $H(s)$  may have time delay included, in the form of  $e^{-sT_0}$ , where  $T_0 > 0$ .

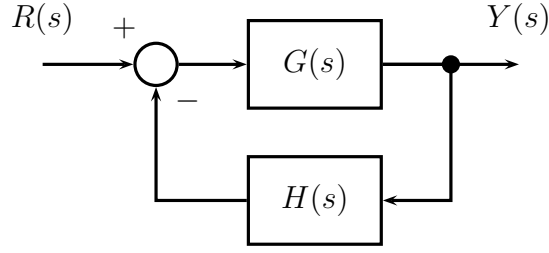


Fig. 1. Considered closed-loop system

Suppose it holds that:  $G(s) = kG_1(s)e^{-sT_0}$  (with a proper  $G_1(s)$ ,  $k > 0$ ), then the closed-loop characteristic equation has the form

$$\begin{aligned} 1 + G(s)H(s) &= 0, \\ 1 + kG_1(s)H(s)e^{-sT_0} &= 0. \end{aligned}$$

As can be seen, it is impossible to derive the destabilizing value of  $k$  since the polynomial expression has not been obtained.

To overcome this problem, the standard Padé approximation will be compared with the generalized Padé approximation presented in [14], to obtain the information which is crucial from the stability analysis viewpoint, i.e. whether the generalized approximation is an appropriate tool for stability analysis of time-delayed systems.

The question is given rise to by the statements of the authors of [14], who claim that their approximation is suitable for time-domain analysis of the systems with time delays, but omit the problem of stability analysis, except for approximation of time responses.

### 3. SYMMETRICAL GENERALIZED PADÉ APPROXIMATION

Assume that the symmetrical approximation of the dead-time transfer function is of the following form:

$$G_{SGP}(s) = \frac{d_0 - d_1(sT_0)^a + d_2(sT_0)^{2a} + \dots + (-1)^n d_n(sT_0)^{na}}{d_0 + d_1(sT_0)^a + d_2(sT_0)^{2a} + \dots + d_n n(sT_0)^{na}}, \quad (12)$$

where  $0 < a \leq 2$  corresponds to a fractional-order.

At first, let the fitting polynomial be introduced [14]

$$P_{a,r}(T_0, s) = \sum_{i=0}^r c_i (sT_0)^{ia} \quad (13)$$

and the fitting error function ( $r \geq 2n + 1$ )

$$\epsilon(j\omega) = \sigma e^{j\theta} = P_{a,r}(T_0, j\omega) - e^{jT_0\omega}, \quad (14)$$

where  $\sigma \geq 0$ ,  $\theta \in \mathcal{R}$ .

In order to evaluate the quality of fitting of the polynomial to the exponential function in the frequency range  $\omega_1 \leq \omega \leq \omega_2$ , the following optimization cost function is introduced after [14]:

$$f(\underline{c}) = \int_{\omega_1}^{\omega_2} (\lambda_1 \sigma + \lambda_2 |\theta|) d\omega, \quad (15)$$

where  $\underline{c} = [c_0, c_1, c_2, \dots, c_r]^T \in \mathcal{R}^{r+1}$  is the vector of the coefficients of the fitting polynomial.

The optimal vector  $\underline{c}^*$  is the result of the solution of the following optimization task:

$$\underline{c}^* = \arg \min_{c_i \in \mathcal{R}, i \in \{0, 1, 2, \dots, r\}} f(\underline{c}). \quad (16)$$

On the basis of the procedure presented in [14], when optimal  $\underline{c}$  has been obtained, it must hold that

$$\begin{bmatrix} c_{n+1} & c_n & \cdots & c_1 \\ c_{n+2} & c_{n+1} & \cdots & c_2 \\ \vdots & \vdots & \ddots & \vdots \\ c_{2n+1} & c_{2n} & \cdots & c_{n+1} \end{bmatrix} \begin{bmatrix} d_0 \\ -d_1 \\ \vdots \\ (-1)^n d_n \end{bmatrix} = \underline{0}, \quad (17)$$

where  $c_i$  are the elements of the optimal coefficient vector.

The obtained approximation of the dead time will be used in FOMCON toolbox [13] to obtain the values of the stability margins of the closed-loop system for various values of  $a$ .

#### 4. OPTIMIZATION PROCEDURE

The selected optimization procedure must be applicable in the task of finding the minimum of a function  $f : \mathcal{R}^{r+1} \rightarrow \mathcal{R}$ , where  $r + 1$  is a number of coefficients of the fitting polynomial. The cost function  $f$  is expected to have a local minimum or local minima, and the argument  $\underline{c}$  of  $f$  can be thought of as a vector of coefficients of the fitting polynomial.

The BFGS (Broyden-Fletcher-Goldfarb-Shanno) method is chosen, since [2]:

- it has properties of quasi-Newton methods,
- is has properties of conjugate gradient methods,
- it preserves the DFP (Davidon-Fletcher-Powell) method property, to maintain positive definiteness of a Hessian,
- line search in a direction can be performed approximately,
- the latter allows one to shorten the time duration of the single iteration,
- the method is more effective in comparison with DFP algorithm.

Since this method is an approximation of the Newton method, the calculation of the gradient  $\nabla f(\underline{c}) : \mathcal{R}^{r+1} \rightarrow \mathcal{R}^{r+1}$  can be also approximated.

For given  $\delta \ll 1$  and a versor

$$\underline{e}_i = \begin{cases} e_j = 1 & \text{for } j = i \\ e_j = 0 & \text{for } j \neq i \end{cases}$$

the approximation of a gradient  $\nabla f(\underline{c})$  with  $\underline{c} \in \mathcal{R}^{r+1}$  is obtained by evaluation of the aim function  $f$  at  $2r + 1$  points, using the following central difference formula for the  $i$ -th entry of the gradient vector ( $i = 1, \dots, r + 1$ ):

$$(\nabla f(\underline{c}))_i = \frac{f(\underline{c} + \delta \underline{e}_i) - f(\underline{c} - \delta \underline{e}_i)}{2\delta}, \quad (18)$$

where  $\delta$  is a sufficiently small number, changing  $i$ -th entry of  $\underline{c}$ .

The complete algorithm of the BFGS method is as follows [2]:

- 1) put  $k = 0$ , and define  $\underline{c}^{(0)}$  and the initial approximate inverse of the Hessian  $\mathbf{H}_0^{-1}$  ( $\mathbf{H}_0^{-1} = \mathbf{H}_0^{-T} > 0$ ),
- 2) calculate  $\nabla f(\underline{c}^{(k)})$  and  $\underline{d}^{(k)} = -\mathbf{H}_k^{-1} \nabla f(\underline{c}^{(k)})$ ,
- 3) select the length  $\alpha_k$  of the step

$$\alpha_k = \arg \min_{\alpha} f(\underline{c}^{(k)} + \alpha \underline{d}^{(k)}),$$

and improve the current solution  $\underline{c}^{(k+1)} = \underline{c}^{(k)} + \alpha_k \underline{d}^{(k)}$ ,

- 4) calculate:  $\nabla f(\underline{c}^{(k+1)})$ ,  $\Delta \underline{c}^{(k)} = \alpha_k \underline{d}^{(k)}$ ,
- 5) calculate the change in the gradient  $\Delta \nabla f(\underline{c}^{(k)}) = \nabla f(\underline{c}^{(k+1)}) - \nabla f(\underline{c}^{(k)})$ ; and check the stopping criterion – if satisfied terminate the algorithm, otherwise, proceed to Step 6),
- 6) improve the approximate of the inverse of the Hessian (Sherman-Morrison formula)

$$\begin{aligned} a &= (\Delta \underline{c}^{(k)})^T \Delta \nabla f(\underline{c}^{(k)}), \\ \mathbf{B} &= \Delta \underline{c}^{(k)} (\Delta \nabla f(\underline{c}^{(k)}))^T, \\ \mathbf{H}_{k+1}^{-1} &= \mathbf{H}_k^{-1} + \frac{a + (\Delta \nabla f(\underline{c}^{(k)}))^T \mathbf{H}_k^{-1} \Delta \nabla f(\underline{c}^{(k)})}{a^2} \Delta \underline{c}^{(k)} (\Delta \underline{c}^{(k)})^T + \\ &\quad - \frac{\mathbf{H}_k^{-1} \mathbf{B}^T + \mathbf{B} \mathbf{H}_k^{-1}}{a} \end{aligned}$$

- 7) put  $k := k + 1$  and go to Step 2.

The choice of  $\alpha_k$  forces the minimum of the cost function to be sought in a direction, e.g., at 10 linearly-spaced points with  $\alpha \in \{0.1, 0.2, \dots, 1\}$ , choosing the optimal value amongst them, to reduce the computational burden.

In order to perform the optimization, the following have been assumed:

- $\omega_1 = 0.001 \frac{\text{rad}}{\text{sec}}$ ,
- $\omega_2 = 10 \frac{\text{rad}}{\text{sec}}$ ,
- $\lambda_1 = 0.9$ ,
- $\lambda_2 = 1$ ,
- $\delta = 0.01$ ,
- stopping criterion for BFGS method:  $\|\nabla f(\underline{c}^{(k)})\| \leq 0.05$  or 100 iterations exceeded,
- initial solution  $c_i^{(0)} = \frac{1}{\Gamma(a(i-1)+1)}$  ( $i = 1, 2, \dots, r + 1$ ).

## 5. RESULTS

The system taken into consideration has the following transfer functions:

$$G(s) = \frac{10}{20s^2 + 15s + 1} e^{-0.5s},$$

$$H(s) = 1,$$

where exact (true) values of the stability margins for the fully integer-order system are:

$$\lambda = 10.0456 \text{ dB at } \omega = 1.1722 \frac{\text{rad}}{\text{sec}},$$

$$\Delta\varphi = 41.5361 \text{ rad at } \omega = 0.5633 \frac{\text{rad}}{\text{sec}}.$$

The values obtained for  $a = 1$  from numerical calculations using symmetrical approximation and FOMCON toolbox are (see Fig. 3):

$$\hat{\lambda}\Big|_{a=1} = 10.2796 \text{ dB},$$

$$\hat{\Delta\varphi}\Big|_{a=1} = 41.6417 \text{ rad}.$$

Figure 2 presents coefficients of a fitting polynomial with  $r = 2$ , and Figure 3 presents the values of gain and phase margins for various values of  $a$ .

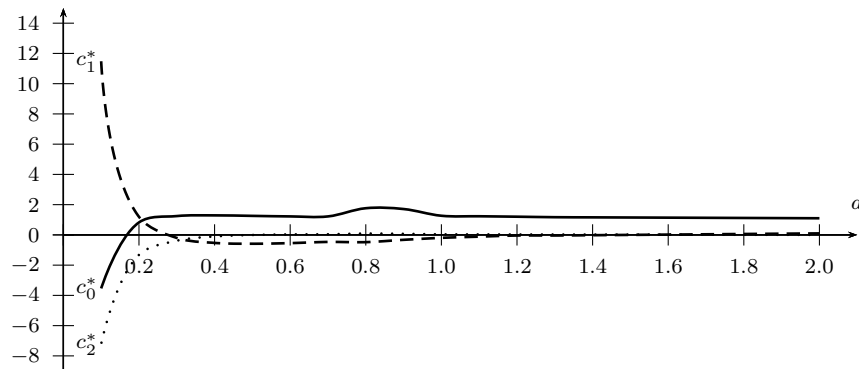
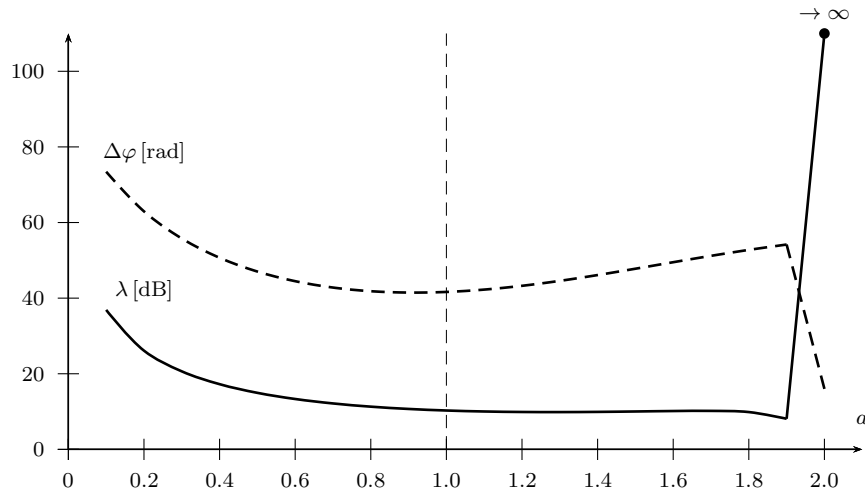


Fig. 2. Coefficients of a fitting polynomial with  $r = 2$

On the contrary to the statements of the paper [14], where its authors claimed the generalised approximation is suitable to use in approximating the time responses of time-delayed systems, it is obvious that only for the case  $a = 1$ , when the generalized approximations yields the standard Padé approximation, the results are accurate. When  $a = 1$ , the stability margins obtained from Figure 3, match the true ones obtained from FOMCON toolbox.

The changes in phase and gain margins depicted in Figure 3 are the result of changing the slopes in the Bode plots of the open-loop transfer function whenever  $a \neq 1$ , and thus, changing the information concerning stability border.

Fig. 3. Stability margins vs.  $a$ 

## 6. SUMMARY

The discussed method of constructing a generalised Padé approximation proves to be unsuitable for analysis of stability margins of closed-loop systems with time delays in any of the loops, though as reported in [14], it is useful in determining the shape of the time response of time-delayed systems, where standard (integer-order) Padé-approximated transfer functions fails to present accurate results (see [14]). The analysis of the use of the generalised approximation, followed by the conclusions are the novelty of this paper, since this approach has been hitherto concentrated on time responses only.

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## ABSTRACT

The paper discusses the applicability of the generalized Padé approximation to stability analysis of systems with time delays, which proves to be inapplicable. As the result of the approximation, the stability margin of the resulting fractional-order closed-loop system is strongly dependent on the order of the fractional-order approximation, leading to inaccuracies.

## ANALIZA UOGÓLNIONEJ APROKSYMACJI PADÉGO DO OCENY ZAPASU STABILNOŚCI

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W artykule przedstawiono dyskusję możliwości zastosowania uogólnionej aproksymacji Padégo w analizie stabilności układów regulacji zawierających opóźnienia transportowe, w wyniku czego stwierdzono niecelowość stosowania metody. Na skutek aproksymacji, zapas stabilności otrzymanego zamkniętego układu ułamkowego rzędu silnie zależy od ułamkowego rzędu transmitancji aproksymującej opóźnienie, co prowadzi do niedokładności.

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