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BODE PLOT BASED STABILITY ANALYSIS OF A FRACTIONAL-ORDER PI CONTROLLER WITH FIRST-ORDER TIME-DELAYED PLANT

Keywords: Stability analysis, fractional-order PI controller, time-delay, Bode plot

1. INTRODUCTION

A classical PID controller is one of the most common tools used in industrial automation. This is due to its simplicity and robustness for parameter uncertainty [9, 10, 12, 13]. However, the increase in computing power of processors has recently led to better analysis of fractional-order systems. In many papers it was proven that the non-integer order PID controller, to be exact the $PI^\lambda D^\mu$ controller, provides lower quality indices which leads to conclusion that tracking performance of the closed-loop system will be better. In author's previous work [4, 11, 12] it was shown in simulation and experimental results that the systems with fractional-order PI controllers give lower overshoot and lower values of quality indices IAE and ISE. Applications of fractional-order system in control theory field has become in-depth in recent years and are described in books, such as [9, 8]. The novelty of this paper is to tune a fractional-order PI controller with respect to Bode gain and phase margins which are an important aspects during design of control systems. In typical control systems, the phase margin is between $30°$ and $60°$, while the gain margin is between $5\,\mathrm{dB}$ and $10\,\mathrm{dB}$. The paper is organized as follows: Section 2 describes the problem stated in this paper and how to tune a fractional-order PI controller with respect to phase margin. Section 3 presents the simulation results of the stability analysis, and the last section provides the conclusions and direction of author's further research.

2. PROBLEM STATEMENT

2.1. STRUCTURE OF THE CONTROL SYSTEM

The forward loop has a first-order linear model of Inteco Modular Servo [7, 6] that is described by transfer function:

$$G(s) = \frac{b_0}{a_1 s + 1} e^{-sL}, \tag{1}$$

where [6]:

$$b_0 = 169.20, a_1 = 1.065, L = 0.2\,\mathrm{sec}. \tag{2}$$

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The time delay L has been added in this paper, to enable analysis of the system ensuring certain level of stability margins. The actual Inteco Modular Servo has a nonlinear characteristics, and the model (1) with additionally introduced time delay is valid when this characteristics is compensated by a serial precompensator introduced to the real open-loop system [6].

The fractional-order PI^λ controller is described by the transfer function:

$$C(s) = K_p + \frac{K_i}{s^\lambda}, \tag{3}$$

where λ refers to order of the integral term (usually $0 < \lambda \leqslant 1$) [2, 5, 3].

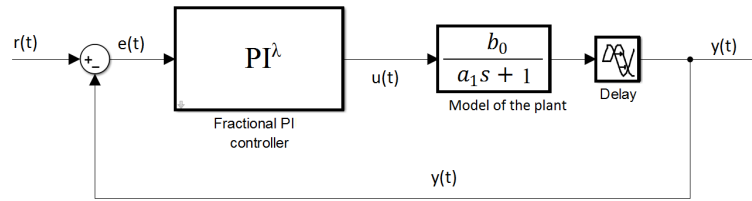The block diagram of the closed-loop system is shown in Figure 1

Fig. 1. Block diagram of the considered control system (model refers to servo with added time delay) and fractional-order PI controller

In the forward path there is a gain and phase margin tester in form $Ae^{-j\phi}$, where A and ϕ are the gain, and phase margins, respectively. The tester is not added to the simulation model, it is used only for tuning of the fractional-order PI controller. The controller parameters are chosen in the specified form to ensure certain gain and phase margins

2.2. TUNING RULES

The quasi-polynomial describing the closed-loop characteristic equation of the system shown in Figure 1 with $C(s)$ given by (4) is:

$$\delta(s) = Ae^{-j\phi} \left(b_0 K_p + \frac{b_0 K_i}{s^\lambda} \right) e^{-sL} + a_1 s + 1. \tag{4}$$

Let $\delta^*(s)$ be described by the following equation assuming $s = j\omega$:

$$\delta^*(j\omega) = \delta_r^*(\omega) + j\delta_i^*(\omega). \tag{5}$$

It is necessary to rewrite the $\delta(s)$ quasi-polynomial as

$$\delta^*(s) = Ae^{-j\phi} b_0 K_p s^\lambda + Ae^{-j\phi} b_0 K_i + (a_1 s + 1) e^{sL} s^\lambda. \tag{6}$$

To obtain the range of stabilizing parameters (K_p, K_i), it is important to apply the following two Theorems, namely, of *Hermite-Biehler* and *Pontryagin* to assure that the roots are real and interlaced. More comprehensive description of the Theorems can be found in [5, 4, 12], and [1, 2] and it is omitted here for an undisturbed presentation of new results.

The parameter K_p can be described by equation:

$$K_p = \frac{-1}{Ab_0 \sin\left(\left(\frac{\pi}{2}\right)\lambda\right)} \left[a_1\omega\sin\left(\lambda\right) + \omega L + \phi\right] + \sin\left(\frac{\pi}{2} + \omega L + \phi\right) \qquad (7)$$

In accordance with [5], the range of parameters K_i ensuring closed-loop stability based on BIBO criteria must meet the conditions

$$K_i = \frac{1}{Ab_0 \sin\left(\left(\frac{\pi}{2}\right)\lambda\right)} \left[a_1\omega^\lambda\sin\left(\omega L + \phi\right)\right] + \omega^\lambda\sin\left(\omega L\right) \qquad (8)$$

Since $\delta_i^*\left(\omega\right)$ is an odd function, it has root at $\omega = 0$, thus for $\omega = \omega_0 = 0$:

$$\delta_i^*\left(\omega\right) = b_0 K_p + 1. \qquad (9)$$

The range of values K_p and K_i that fulfill the conditions for BIBO stability of the closed-loop system are presented in Section 3. The paper is focused on obtaining ranges of controller's parameters, K_p and K_i that ensures Bounded-Input-Bounded-Output (BIBO) stability on simulation-based results.

2.3. STABILITY CRITERIA

To present the stability surface understood as the range of all (K_p, K_i) for which the closed-loop system is stable, authors defines two terms. The criterion is BIBO-based, and is composed of the two conditions:

- verify if simulation time counter is the same as the desired length of simulation,
- if the above horizons are equal – verify if the output signal from the closed-loop system diverges, either exponentially or in the oscillatory manner.

If any of the two above conditions occur, the closed-loop system is considered to be unstable. Possible unstable behavior of the closed-loop system will be the result of large gains in P or I branches accompanied by the existence of the time-delay in the open-loop transfer function.

3. SIMULATION RESULTS

The stability analysis was performed for a range of ω in $(0, 5\pi)$ with step $\frac{\pi}{20}$. The range of parameters K_p and K_i are calculated by using (7) and (8). The λ parameter takes values in $(0.1, 1)$ with step 0.1 and simulation time set to 35 sec, and the gain margin S equal to 1 and phase margin in the range $(0, 60°)$ with step of $15°$. The first-order plant parameters are as in (2). Experiments have been performed with the Oustaloup-Recursive-Approximation used in FOMCON toolbox [14] for Fractional Order integrator in the Matlab environment. The following approximation parameters of fractional-order integrator was chosen: the order was equal to 5, and frequency range was $(0.001, 1000)$ rad/sec, and the sampling period for continuous-time approximation of fractional-order integration was 0.01 sec.

Figures 2-5 present the stability region for changing order of the integral part with constant phase margin.

In comparison to author's previous paper where the phase margin was not taken into consideration during the tuning of the fractional-order PI controller, the stability surface is
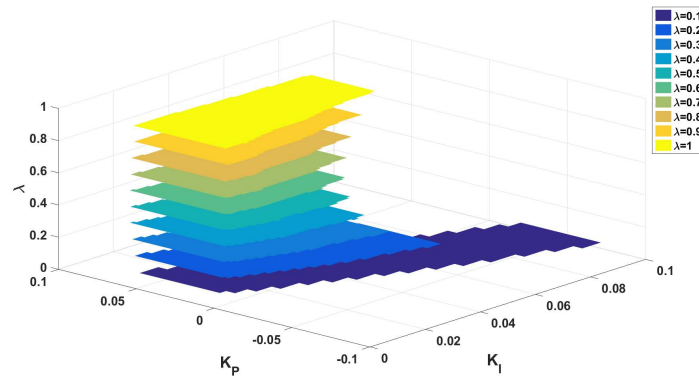
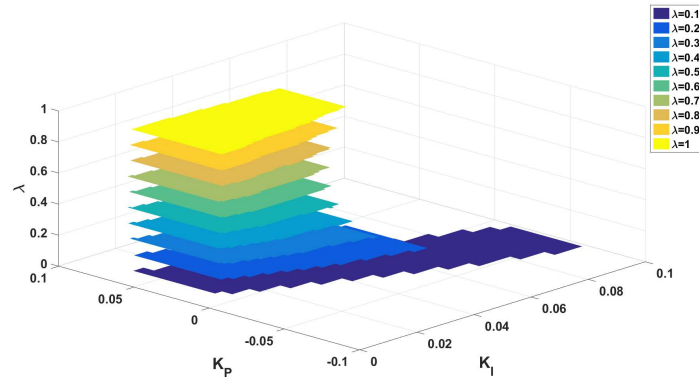Fig. 2. Stability surface of the closed-loop system for phase margin not less than $15°$

Fig. 3. Stability surface of the closed-loop system for phase margin not less than $30°$
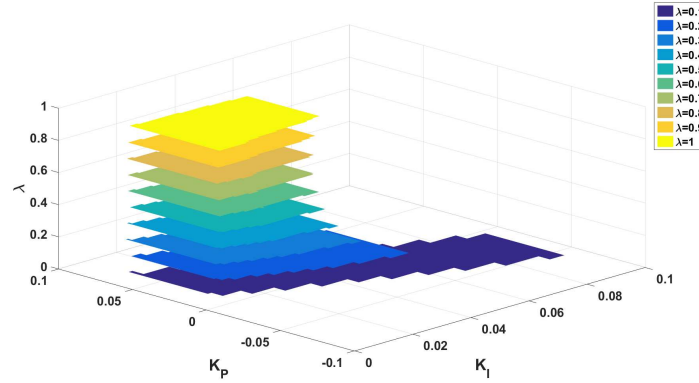

Fig. 4. Stability surface of the closed-loop system for phase margin not less than $45°$
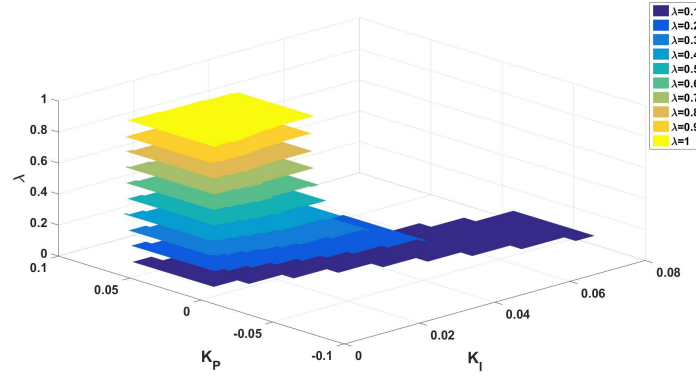
Fig. 5. Stability surface of the closed-loop system for phase margin not less than $60°$

sufficiently smaller. To show the difference, the Figure 6 presents the stability surface of the closed-loop system when the controller parameters are only based on *Hermite-Biehler* and *Pontryagin* theorems without the phase margin tester.
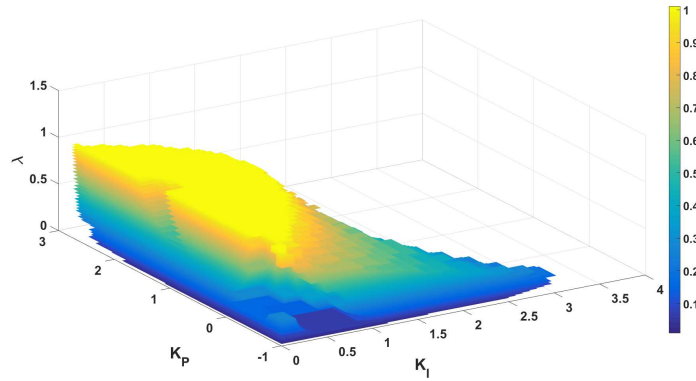

Fig. 6. Stability surface of the closed-loop system without phase margin tester

It is necessary to present the Bode plots to verify the analyzed closed-loop system with phase margin tester. Figures 7 and 8 in closer view, refers to the closed-loop system with controller parameters $K_P = 0.01981$, $K_I = 0.01871$ and fractional-order integrator $\lambda = 0.8$ and phase margin $\phi = 48°$.

As it can be noted from Figures 2-6 the closed-loop system is stable with selected controller parameters, the calculated value of the gain margin $A = 2.2807$, so regarding to the definition of gain margin, if the open-loop gain is multiplied by A, the closed-loop system is on the boundary of the stability region. To prove the alter, Figure 9 presents responses of the closed-loop system with parameters mentioned above vs. the reference signal *r(t)*.
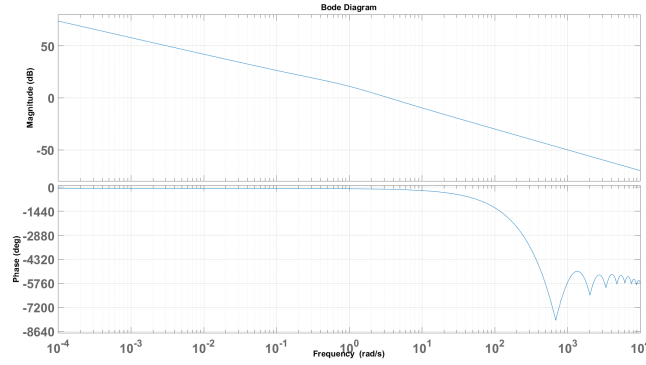
Fig. 7. Bode plots of the closed-loop system with a fractional-order PI controller with phase margin $\phi = 48°$ and gain margin $A = 2.2807$
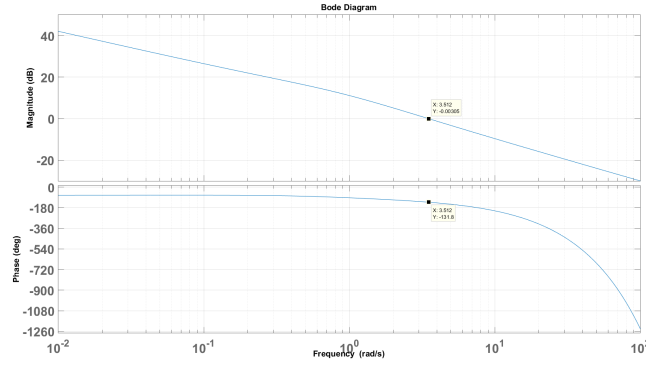

Fig. 8. Closer view of bode plots of the closed-loop system with a fractional-order PI controller

Figure 9 proves that the stability analysis and assumptions taken at the beginning of the article were correct. This can bring new possibilities to analyze fractional-order systems when gain or phase margins are taken into consideration.

4. CONCLUSIONS

The presented results are only examples of the closed-loop system stability analysis. It can be noted that the phase margin tester added in the tuning phase of controller parameters K_p and K_i restricts the stability area, which is an important aspect during the control system design. Moreover, the difference between the stability surface for specific value of λ insignificantly differ from each other for this type of the plant. The negative values of parameter K_p of the controller are correct because in the numerator of the transfer function there is an expression $(-a_1 s^\lambda + 169.20)$ where a_1 is a sufficiently small positive number and for calculating the gain of the forward path when s tends to 0 this expression will be still positive.
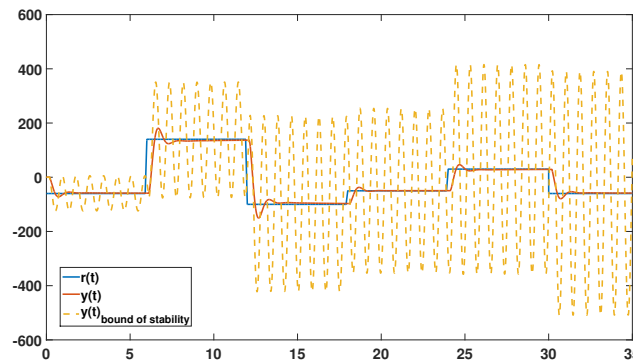
Fig. 9. The responses of the closed-loop system with with phase margin $\phi = 48°$ $y(t)$ and the closed-loop system on the bound of stability $y(t)_{\text{bound of stability}}$

The further research will investigate the tracking performance of the closed-loop system with phase margin tester.

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ABSTRACT

The main aim of the paper is to present the stability analysis of a fractional-order PI controller with first-order, time-delayed plant based on Bode plot. In the paper there is a comparison, presented between the tuning method based on Hermite-Biehler and Pontryagin theorems with, and without taking the gain and phase margins into consideration. The quality performance is based on two quality indices, namely the Integral of Absolute Error and the Integral of a Squared Error.

ANALIZA STABILNOŚCI UKŁAKU Z REGULATOREM RZĘDU UŁAMKOWEGO I OBIEKTEM PIERWSZEGO RZĘDU Z OPÓŹNIENIEM NA PODSTAWIE CHARAKTERYSTYK BODEGO

STRESZCZENIE

Głównym celem artykułu jest analiza stabilności obiektu pierwszego rzędu z opóźnieniem oraz regulatora PI rzędu ułamkowego na podstawie charakterystyk Bode'go. W pracy zaprezentowano porównanie metod strojenia opartych na teoriach Hermite-Biehler'a i Pontryagin"a gdy zapas modułu oraz fazy są brane pod uwagę oraz gdy nie są brane pod uwagę. Jakość śledzenia jest oparta na dwóch wskaźnikach jakości całki z wartości bezwzględniej uchybu (IAE) oraz całki z kwadratu uchybu (ISE).