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A NONLINEAR MODEL FOR TRACKING TURNING TARGETS

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1. INTRODUCTION

Numerous target tracking models have been proposed over the years for the purposes of air vehicle tracking, and comprehensive surveys of the literature are provided in [2] and [7]. The majority of these models are quite deliberately designed to be linear in order to exploit the optimality of the Kalman Filter, although exceptions to this statement are the *Coordinated Turn* model defined in [2], page 205, and the more general *Constant-Turn* model in [7]. Both of the latter two models are designed specifically for kinematic turns in three-dimensional space and it is this form of motion that is of interest in the present paper.

The popular Coordinated Turn model ([3], [8]) is designed to track aircraft turning in a constant plane and can be formulated either in linear form with an estimate of the turn rate obtained from speed and acceleration, or nonlinearly with the turn rate included in the state vector; in the latter case, Extended Kalman Filter methods or similar are required (see, for example, [2], page 209).

In contrast, the intrinsically nonlinear *Constant-Turn* model ([7], section VI) has not, so far as is known, been applied systematically for tracking turning targets, despite its simplicity and generality. The present paper re-derives the fundamental kinematic equations in the context of helical (spiral) target motion (of which turning in the plane is a special case), and demonstrates that nine state vector components are adequate for tracking purposes using a nonlinear filter.

Section 2 below derives the kinematic equations, while Section 3 provides an example of the tracking filter behaviour.

A note on coordinate systems is in order: throughout this paper, the right-handed East-North-Up (ENU) system is defined relative to a location on the earth's surface, with the east and north directions lying in the local horizon plane, north pointing towards true north, and 'up' to complete. The Earth-Centred Rotating system (ECR; sometimes known as Earth-Centred, Earth-Fixed) uses an origin at the earth's centre, x through the Greenwich meridian, z pointing through the north pole and y to complete; this system is fixed to the earth and rotates with it. The Earth-Centred Inertial (ECI) system is fixed relative to the stellar background, shares the same ECR equatorial plane and is defined to be coincident with the ECR system at a defined time (here $t = 0$).

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2. KINEMATIC MODEL GENERATION

A generic helix or spiral form of motion can be created using the following equations in generic Cartesian coordinates:

$$\underline{x} = \underline{x}_0 + \boldsymbol{Q} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}, \tag{1}$$

where vector $\underline{x}_0$ is a constant spatial offset and $\boldsymbol{Q}$ is a rotation matrix from the innate ξ-η-ζ coordinates into the chosen reference frame. Vector $\underline{x}$ can be relative either to ENU or to ECR coordinates, depending on the context.

The ξ, η, ζ helix coordinates are assumed to be given by the set

$$\begin{aligned} \xi &= \lambda \cos \mu t, \\ \eta &= \lambda \sin \mu t, \\ \zeta &= \zeta_0 + Ut, \end{aligned}$$

where λ is the turn radius in the ξ-η plane, μ is the turn rate and U defines a rate of climb or descent relative to the ξ-η plane.

These equations describe circular motion in the ξ-η plane, with the target moving at a constant speed U in the positive ζ direction with fixed offset ζ_0. The net effect is helical, while a purely circular motion in the same plane can be generated by setting $U = 0$. The basic motion relative to the ξ-η-ζ coordinates is illustrated in Figure 1, with values in kilometres.

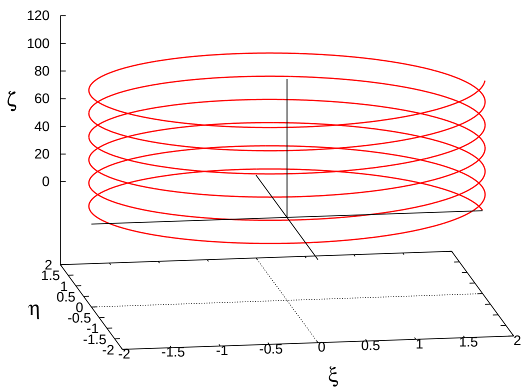

Fig. 1. Helical motion in ξ-η-ζ coordinates

Here, the orbit radius is 2 km and the vertical speed (in the ζ direction) set at 200 ms^{-1}.

The orientation of the helix with respect to the ENU (or ECR) coordinate axes is conveniently given by the constant rotation matrix $\boldsymbol{Q}$:

$$\boldsymbol{Q} = \begin{bmatrix} \cos\Omega & -\sin\Omega\cos i & \sin\Omega\sin i \\ \sin\Omega & \cos\Omega\cos i & -\cos\Omega\sin i \\ 0 & \sin i & \cos i \end{bmatrix},$$

deliberately borrowing perifocal-type coordinates Ω and i from [1]. Here, Ω is the analogue of the longitude of ascending node in the ENU x-y plane, while i defines an inclination angle relative to the ENU z-axis*. Following [1], page 80, and using the ENU frame as an example, these angles are illustrated in Figure 2, where the vector $\underline{\xi}$ stands for an arbitrary direction in the ξ-η-ζ frame.

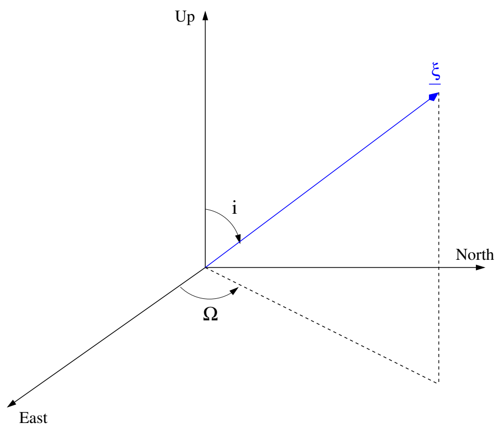

Fig. 2. Relation between ξ-η-ζ coordinates and ENU

If it is assumed that the helix orientation does not change with time, matrix $\boldsymbol{Q}$ will be a constant. Therefore, the ENU (or ECR) velocity and acceleration vectors will be given by:

$$\underline{\dot{x}} = \boldsymbol{Q} \begin{bmatrix} -\lambda\mu\sin\mu t \\ \lambda\mu\cos\mu t \\ U \end{bmatrix}, \tag{2}$$

and

$$\underline{\ddot{x}} = \boldsymbol{Q} \begin{bmatrix} -\lambda\mu^2\cos\mu t \\ -\lambda\mu^2\sin\mu t \\ 0 \end{bmatrix}, \tag{3}$$

a superscript dot indicating time derivative.

*The analogue of the argument of perigee is ignored, since the ξ and η coordinates can be chosen arbitrarily within the plane of the motion.

Now, the unit ENU helix orientation vector defining its axis is given simply by

$$\underline{\hat{q}} = \begin{bmatrix} \sin\Omega\sin i \\ -\cos\Omega\sin i \\ \cos i \end{bmatrix}, \tag{4}$$

and it is a straightforward matter to show that

$$\begin{aligned} \underline{\hat{q}}^T \underline{\dot{x}} &= U, \\ \underline{\hat{q}}^T \underline{\ddot{x}} &= 0. \end{aligned}$$

That is, the component of acceleration in the direction of $\underline{\hat{q}}$ is zero*, a fact that will become useful later in the analysis.

Given $\underline{\hat{q}}$, two further valuable relations follow from equations (2) and (3):

$$|\underline{\hat{q}} \times \underline{\dot{x}}| = \lambda\mu, \tag{5}$$

$$|\underline{\hat{q}} \times \underline{\ddot{x}}| = \lambda\mu^2, \tag{6}$$

which can be used to derive the radius of the turn λ and the turn rate parameter μ.

Before proceeding to the extraction of generic kinematic equations in Section 2.2, it is worth first comparing the Coordinated Turn model to this form of helical target motion.

2.1. Comparison With The Coordinated Turn Model

The kinematic equation for the Coordinated Turn model is given by the following ([2], page 205):

$$\frac{d}{dt}\underline{\ddot{x}} = -\mu^2 \underline{\dot{x}}, \tag{7}$$

for turn rate μ, here ignoring process noise.

If, however, the actual motion is governed by equations (2) and (3), it is straightforward to show that

$$\frac{d}{dt}\underline{\ddot{x}} = -\mu^2 \underline{\dot{x}} + \mu^2 \boldsymbol{Q} \begin{bmatrix} 0 \\ 0 \\ U \end{bmatrix}.$$

This is very nearly of the form of equation (7), apart from the final constant acceleration rate vector, which would require an additional three state components in the tracking filter. For circular motion, however, for which $U = 0$, the Coordinated Turn model is appropriate as it stands.

Note that the Coordinated Turn model is only linear if μ is known or can be estimated; [2] provides suitable equations when μ is incorporated into the state vector.

*This is readily appreciated by considering the nature of the motion in the ξ-η-ζ coordinates.

2.2. Generic Kinematic Equations

This section derives a more generally-applicable set of kinematic equations for helical type motions in the ECR (or, if desired, ENU) coordinate system. Returning to Section 2, consider the two equations (2) and (3):

$$\underline{\dot{x}} = \boldsymbol{Q} \begin{bmatrix} -\lambda\mu \sin \mu t \\ \lambda\mu \cos \mu t \\ U \end{bmatrix}, \tag{8}$$

$$\underline{\ddot{x}} = \boldsymbol{Q} \begin{bmatrix} -\lambda\mu^2 \cos \mu t \\ -\lambda\mu^2 \sin \mu t \\ 0 \end{bmatrix}, \tag{9}$$

with

$$\boldsymbol{Q} = \begin{bmatrix} \cos\Omega & -\sin\Omega\cos i & \sin\Omega\sin i \\ \sin\Omega & \cos\Omega\cos i & -\cos\Omega\sin i \\ 0 & \sin i & \cos i \end{bmatrix}.$$

Here, $\boldsymbol{Q}$ is regarded as a rotation matrix from the ξ-η-ζ coordinate system to ECR, rather than to ENU. The constant offset $\underline{x}_0$ in equation (1) has zero time derivative.

Reverse equation (8), so that

$$\begin{bmatrix} -\lambda\mu \sin \mu t \\ \lambda\mu \cos \mu t \\ U \end{bmatrix} = \boldsymbol{Q}^{-1} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}, \tag{10}$$

using a fairly obvious notation for $\underline{\dot{x}}$, and where

$$\boldsymbol{Q}^{-1} \equiv \boldsymbol{Q}^T = \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega\cos i & \cos\Omega\cos i & \sin i \\ \sin\Omega\sin i & -\cos\Omega\sin i & \cos i \end{bmatrix},$$

(the superscript ^{T} stands for matrix transpose). Therefore, multiplying out and changing the sign on equation (10):

$$\begin{aligned} \lambda\mu \sin \mu t &= -\dot{x}\cos\Omega - \dot{y}\sin\Omega, \\ \lambda\mu \cos \mu t &= -\dot{x}\sin\Omega\cos i + \dot{y}\cos\Omega\cos i + \dot{z}\sin i, \\ U &= \dot{x}\sin\Omega\sin i - \dot{y}\cos\Omega\sin i + \dot{z}\cos i. \end{aligned}$$

The quantities $\lambda\mu \sin \mu t$ and $\lambda\mu \cos \mu t$ can now be substituted into equation (9), so that

$$\begin{aligned} \underline{\ddot{x}} &= -\mu\boldsymbol{Q} \begin{bmatrix} \lambda\mu \cos \mu t \\ \lambda\mu \sin \mu t \\ 0 \end{bmatrix} \\ &= -\mu\boldsymbol{Q} \begin{bmatrix} -\dot{x}\sin\Omega\cos i + \dot{y}\cos\Omega\cos i + \dot{z}\sin i \\ -\dot{x}\cos\Omega - \dot{y}\sin\Omega \\ 0 \end{bmatrix}. \end{aligned}$$

Carrying out the matrix multiplication and simplifying:

$$
\begin{aligned}
\ddot{x} &= -\mu\dot{y}\cos i - \mu\dot{z}\cos\Omega\sin i,\\
\ddot{y} &= \mu\dot{x}\cos i - \mu\dot{z}\sin\Omega\sin i,\\
\ddot{z} &= \mu\dot{x}\cos\Omega\sin i + \mu\dot{y}\sin\Omega\sin i,
\end{aligned}
$$

which relate the ECR (or ENU) acceleration components to the velocity components. Further simplifications can be made if three constants are introduced:

$$\alpha = -\mu\cos i, \tag{11}$$

$$\beta = -\mu\cos\Omega\sin i, \tag{12}$$

$$\gamma = -\mu\sin\Omega\sin i, \tag{13}$$

which provide the following set of kinematic equations for the turning target:

$$\ddot{x} = \alpha\,\dot{y} + \beta\,\dot{z}, \tag{14}$$

$$\ddot{y} = -\alpha\,\dot{x} + \gamma\,\dot{z}, \tag{15}$$

$$\ddot{z} = -\beta\,\dot{x} - \gamma\,\dot{y}. \tag{16}$$

Note that for the helix motion, the *aerodynamic parameters* α, β and γ are constants; they involve the turn rate μ as well as the helix orientation angles Ω and i.

Thus, if α, β and γ are tracked as part of the state vector, the turn rate μ and the orientation angles Ω and i can be obtained from the following relations:

$$
\begin{aligned}
\mu &= \pm\sqrt{\alpha^2+\beta^2+\gamma^2},\\
\tan\Omega &= \tfrac{\gamma}{\beta},\\
\tan i &= \pm\tfrac{\sqrt{\beta^2+\gamma^2}}{\alpha}.
\end{aligned}
$$

Several points are worth bringing out here:

- Equations (14), (15) and (16) cannot be reversed. That is, given the velocity and acceleration components at a single time, it is not possible to algebraically derive α, β and γ. This can be seen by writing equations (14), (15) and (16) in the form

  $$\begin{bmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{bmatrix} = \boldsymbol{M}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}, \quad \text{for} \quad \boldsymbol{M} = \begin{bmatrix}\dot{y} & \dot{z} & 0\\ -\dot{x} & 0 & \dot{z}\\ 0 & -\dot{x} & -\dot{y}\end{bmatrix},$$

  and noting that $\boldsymbol{M}$ is singular.

  However, given rates of change of acceleration in addition to velocity and acceleration components, it is possible to derive unique values of α, β, γ, as is discussed in Appendix A on the model observability.
- The equations imply a target flying at constant ECR speed, since $\dot{x}\ddot{x} + \dot{y}\ddot{y} + \dot{z}\ddot{z} = 0$.

- For constant α, β and γ, exact solutions are possible, since

$$\frac{d^4x}{dt^4} = -\mu^2 \frac{d^2x}{dt^2},$$

and similarly for y and z. These imply purely sinusoidal acceleration components.

- It is possible to express equations (14), (15) and (16) in a 'turn vector' form by writing them as follows:

$$\underline{\ddot{x}} = \alpha \begin{bmatrix} \dot{y} \\ -\dot{x} \\ 0 \end{bmatrix} + \beta \begin{bmatrix} \dot{z} \\ 0 \\ -\dot{x} \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ \dot{z} \\ -\dot{y} \end{bmatrix},$$

noting that the three vectors on the right-hand-side are each normal to the velocity, although not normal to each other.

- The above kinematic equations appear overly-simple for a model that claims to be able to track arbitrarily-oriented helices, but an appreciation of the structure can be gained by noting that the helix orientation vector $\underline{\hat{q}}$, given by equation (4), can be written in the form

$$\underline{\hat{q}} = \frac{1}{\mu} \begin{bmatrix} -\gamma \\ \beta \\ -\alpha \end{bmatrix}, \tag{17}$$

so that $\underline{\ddot{x}}^T \underline{\hat{q}} = 0$, as expected.

- The above formulation is simpler than the Coordinated Turn model, which requires tracked acceleration components in addition to a turn rate parameter. On the other hand, the new model is intrinsically nonlinear and should be treated accordingly.
- Equations (14), (15) and (16) may be compared to the similar set in equation (102) in [7], from which it may be determined that the *angular velocity vector* $\mathbf{\Omega}$ used there is equivalent to vector $\underline{q}$ in the present paper. Therefore, equations (14), (15), (16) can also be written in the form $\underline{\ddot{x}} = \underline{q} \times \underline{\dot{x}}$.
- The units of α, β and γ are inverse time, with expected magnitudes of the order of acceleration divided by speed.

3. TRACKED EXAMPLE

The implementation of equations (14), (15) and (16) into a tracking filter is straightforward. In 'variant' Unscented Kalman Filter (UKF) form (see [4]):

$$\begin{aligned} \dot{x} &= u, \quad \dot{y} = v, \quad \dot{z} = w, \\ \dot{u} &= \alpha\, v + \beta\, w + n_x, \\ \dot{v} &= -\alpha\, u + \gamma\, w + n_y, \\ \dot{w} &= -\beta\, u - \gamma\, v + n_z, \\ \dot{\alpha} &= n_\alpha, \quad \dot{\beta} = n_\beta, \quad \dot{\gamma} = n_\gamma, \end{aligned} \tag{18}$$

where the $n_x, \ldots, n_\gamma$ quantities stand for zero-mean, normally-distributed random values with the appropriate standard deviations. There are, thus, nine state components, resulting in nineteen sigma points. Numerical state vector propagation can be achieved straightforwardly by means of fourth-order Runge-Kutta methods, using time sub-steps of 0.01 s.

The above equations were then applied to the case of a single target undergoing a constant helical, or corkscrew, type of motion about an otherwise constant 2 km-altitude main trajectory, with a fixed ECR speed of 200 ms^{-1}. Measurements from a single ground-based radar were simulated at a 1 Hz data rate, using range uncertainties of 10 m and azimuth and elevation uncertainties of 4 mrad and 1 mrad respectively. Sensor-target ranges varied from 3 km at the start to about 10 km at the end, while the target accelerations involved were of the order of 2 g in magnitude.

The track was initiated from two asynchronous simulated radar measurements, with an initial acceleration uncertainty of 10 ms^{-2}. The initial values of α, β and γ were set to zero with accompanying initial uncertainties of 0.02, and subsequent process noise values were set to 2 ms^{-2} for acceleration and 0.003 for the aerodynamic parameters.

The resulting tracked α-β-γ parameters are shown in Figure 3, inclusive of error bars to indicate associated one-sigma uncertainties.

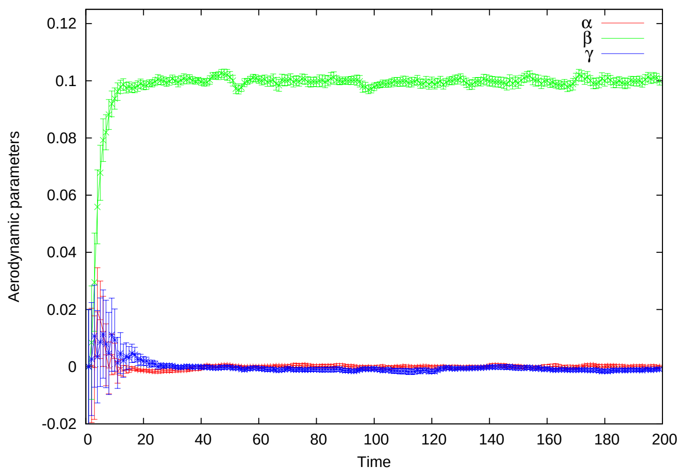

Fig. 3. Lateral helix data, tracked α, β, γ parameters

It can be seen that convergence of α, β and γ occurs rapidly, within about 20 s of track initiation.

Note that, as mentioned on page 11, $\boldsymbol{Q}$ is here regarded as a rotation from the ξ-η-ζ coordinates to ECR, not to ENU. Thus, the helix axial unit vector $\underline{\hat{q}} \approx [0,\, 1,\, 0]^T$ is aligned with the ECR y-axis, as expected.

Using equation (5), it is now possible to estimate the turn radius λ from $\underline{\hat{q}}$ and the ECR velocity components, giving Figure 4.
The initial estimates of λ are significantly in error, since $\underline{q}$ takes a while to stabilise, so the vertical axis in the figure has been truncated for clarity. From about 10 s onwards, λ is close to the correct value of 2 km.

For completeness, the net (Root-Mean-Square) track position and velocity uncertainties are shown in Figure 5.

The 6 degree-of-freedom χ^2 error measure is provided in Figure 6.

Here, the χ^2 quantity is defined as follows:

$$\chi^2 = \left(\underline{x} - \underline{\hat{x}}\right)^T \boldsymbol{P}^{-1} \left(\underline{x} - \underline{\hat{x}}\right),$$

where $\underline{x}$ is the true target position and velocity state, $\underline{\hat{x}}$ is the tracked state estimate and $\boldsymbol{P}$ is the 6×6 error covariance matrix. Under 6 degree-of-freedom χ^2 statistics, one expects no more than 1% of values to exceed the 16.81 99% threshold (marked by the dotted line in Figure 6); since this criterion is satisfied, it may be inferred that the tracking model is statistically consistent.

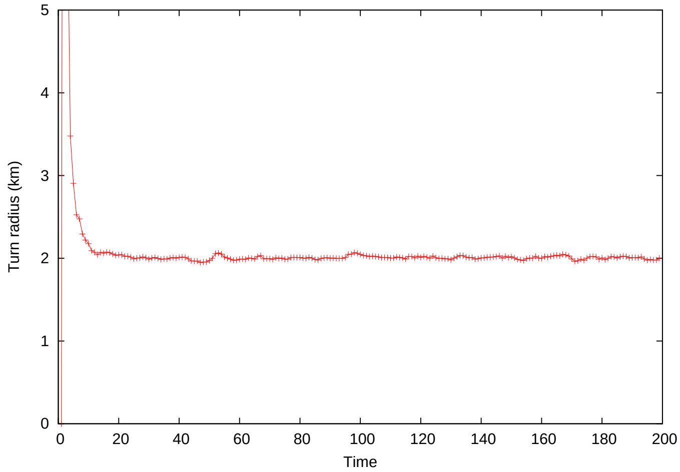

Fig. 4. Lateral helix data, tracked turn radius λ

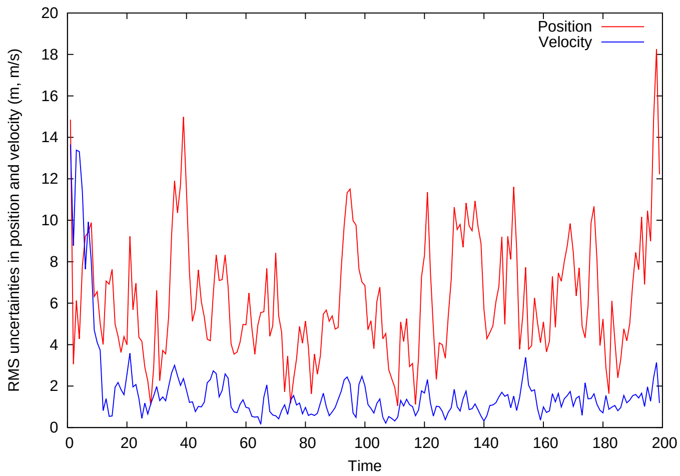

Fig. 5. Lateral helix data, RMS uncertainties

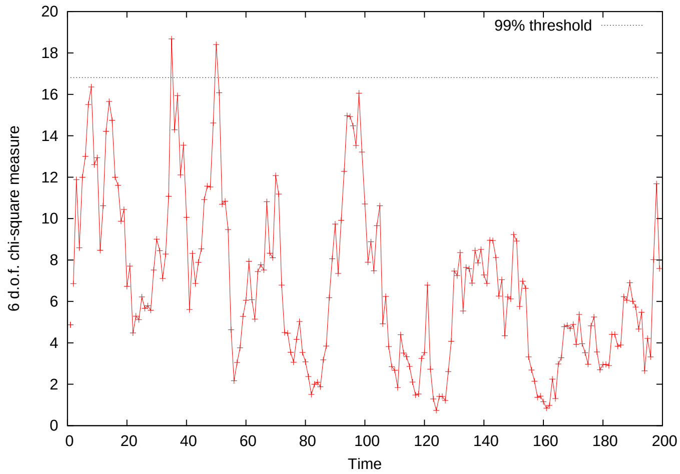

Fig. 6. χ^2 error measure vs time

Note that equation set (3) is formulated in ECR coordinates; if it is desired to track in ECI coordinates, Section 4 below provides the relevant kinematic relations.

4. ECI KINEMATIC EQUATIONS

Equations (14), (15) and (16) provide the kinematic expressions for a turning target in ECR coordinates, whereas in some cases it is necessary to track in ECI coordinates. It is, therefore, necessary to re-express the kinematic equations in ECI form.

The rotation matrix from ECR to ECI is given by:

$$\underline{x}_{eci} = \boldsymbol{R}\underline{x}_{ecr},$$

where

$$\boldsymbol{R} = \begin{bmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and ω is the earth rotation rate.

Therefore,

$$\ddot{\underline{x}}_{eci} = \boldsymbol{R}\ddot{\underline{x}}_{ecr} + 2\dot{\boldsymbol{R}}\dot{\underline{x}}_{ecr} + \ddot{\boldsymbol{R}}\underline{x}_{ecr}, \tag{19}$$

where

$$\dot{\boldsymbol{R}} = -\omega \begin{bmatrix} \sin\omega t & \cos\omega t & 0 \\ -\cos\omega t & \sin\omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \ddot{\boldsymbol{R}} = -\omega^2 \begin{bmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It may be noted here that only the $\ddot{\underline{x}}_{ecr}$ term in equation (19) needs to remain in ECR form, since this will ultimately bring in the aerodynamic parameters. Therefore, noting that

$$\underline{x}_{ecr} = \boldsymbol{R}^T \underline{x}_{eci}, \tag{20}$$

$$\dot{\underline{x}}_{ecr} = \boldsymbol{R}^T \dot{\underline{x}}_{eci} + \dot{\boldsymbol{R}}^T \underline{x}_{eci}, \tag{21}$$

it is possible to write

$$\underline{\ddot{x}}_{eci} = \boldsymbol{R}\underline{\ddot{x}}_{ecr} + 2\dot{\boldsymbol{R}}\left(\boldsymbol{R}^T\underline{\dot{x}}_{eci} + \dot{\boldsymbol{R}}^T\underline{x}_{eci}\right) + \ddot{\boldsymbol{R}}\boldsymbol{R}^T\underline{x}_{eci}.$$

It is not difficult to show that

$$\dot{\boldsymbol{R}}\boldsymbol{R}^T = -\omega\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \dot{\boldsymbol{R}}\dot{\boldsymbol{R}}^T = \omega^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \ddot{\boldsymbol{R}}\boldsymbol{R}^T = -\omega^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so that

$$\underline{\ddot{x}}_{eci} = \boldsymbol{R}\underline{\ddot{x}}_{ecr} - 2\omega\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\underline{\dot{x}}_{eci} + \omega^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\underline{x}_{eci}. \tag{22}$$

Now write the ECR kinematic equations (14), (15), (16) in the matrix form

$$\underline{\ddot{x}}_{ecr} = \boldsymbol{S}\,\underline{\dot{x}}_{ecr}, \tag{23}$$

where matrix $\boldsymbol{S}$ is given by

$$\boldsymbol{S} = \begin{bmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & \gamma \\ -\beta & -\gamma & 0 \end{bmatrix}.$$

Therefore, inserting equations (21) and (23) into equation (22),

$$\underline{\ddot{x}}_{eci} = \left\{\boldsymbol{R}\boldsymbol{S}\boldsymbol{R}^T - 2\omega\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}\underline{\dot{x}}_{eci} + \left\{\boldsymbol{R}\boldsymbol{S}\dot{\boldsymbol{R}}^T + \omega^2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right\}\underline{x}_{eci}.$$

Leaving aside the algebra, the various matrix multiplications are determined to be as follows:

$$\boldsymbol{R}\boldsymbol{S}\boldsymbol{R}^T = \begin{bmatrix} 0 & \alpha & \beta\cos\omega t - \gamma\sin\omega t \\ -\alpha & 0 & \beta\sin\omega t + \gamma\cos\omega t \\ -\beta\cos\omega t + \gamma\sin\omega t & -\beta\sin\omega t - \gamma\cos\omega t & 0 \end{bmatrix},$$

$$\boldsymbol{R}\boldsymbol{S}\dot{\boldsymbol{R}}^T = \omega\begin{bmatrix} -\alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ \beta\sin\omega t + \gamma\cos\omega t & -\beta\cos\omega t + \gamma\sin\omega t & 0 \end{bmatrix}.$$

Combining the various results then gives:

$$\underline{\ddot{x}}_{eci} = \begin{bmatrix} 0 & \alpha - 2\omega & \beta\cos\omega t - \gamma\sin\omega t \\ -\alpha + 2\omega & 0 & \beta\sin\omega t + \gamma\cos\omega t \\ -\beta\cos\omega t + \gamma\sin\omega t & -\beta\sin\omega t - \gamma\cos\omega t & 0 \end{bmatrix}\underline{\dot{x}}_{eci}$$
$$+\omega\begin{bmatrix} -\alpha + \omega & 0 & 0 \\ 0 & -\alpha + \omega & 0 \\ \beta\sin\omega t + \gamma\cos\omega t & -\beta\cos\omega t + \gamma\sin\omega t & 0 \end{bmatrix}\underline{x}_{eci},$$

which are the required kinematic equations when expressed in ECI.

Reverting now to the individual components,

$$\ddot{x} = (\alpha - 2\omega)\,\dot{y} + (\beta\cos\omega t - \gamma\sin\omega t)\,\dot{z} + \omega\,(-\alpha + \omega)\,x, \tag{24}$$

$$\ddot{y} = (-\alpha + 2\omega)\,\dot{x} + (\beta\sin\omega t + \gamma\cos\omega t)\,\dot{z} + \omega\,(-\alpha + \omega)\,y, \tag{25}$$

$$\ddot{z} = (-\beta\cos\omega t + \gamma\sin\omega t)\,(\dot{x} + \omega y) - (\beta\sin\omega t + \gamma\cos\omega t)\,(\dot{y} - \omega x)\,, \tag{26}$$

which are readily expressed in 'variant' UKF form for tracking in ECI coordinates.

It is pointed out that the $\alpha,\ \beta,\ \gamma$ parameters and the corresponding $\underline{q}$-vector remain relative to ECR, so $\underline{q} \cdot \underline{\ddot{x}}_{eci} \neq 0$.

5. CONCLUSIONS

This paper has examined a set of nonlinear kinematic equations designed for tracking air vehicles undergoing helical (spiral) types of motion, and demonstrated the effectiveness of a filter using the 'variant' UKF described in [4]. Although this particular kinematic model was described in [7], it has not (so far as is known) been implemented in tracking filter form (let alone fully nonlinearly). Since the aerodynamic parameters $\alpha,\ \beta,\ \gamma$ are tracked as part of the state vector, the orientation of the helix axis is readily determined, which is potentially of value for situational awareness.

In [7], it is remarked that the correspondence of such trajectories to real-world motions is not clear. However, the use of thrust-vectoring in more agile ground-to-air or air-to-air missiles renders such motions more easily achieved and, indeed, a quick search on the internet shows pictures of exactly such helical behaviours.

ACKNOWLEDGEMENTS

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A. OBSERVABILITY OF THE KINEMATIC MODEL

This section shows that the specified turning target model in Section 2.2 is completely observable, given derivatives of the state position components up to and including the fourth order*. Observability here is in the sense of [5] and [6], which assumes that successive derivatives of the position components (or approximations thereto) can be obtained.

For convenience, write equations (14), (15) and (16) in the form

$$a_x = \alpha v + \beta w, \tag{27}$$

$$a_y = -\alpha u + \gamma w, \tag{28}$$

$$a_z = -\beta u - \gamma v, \tag{29}$$

where $u,\ v,\ w \equiv \dot{x},\ \dot{y},\ \dot{z}$ and the a_x, *etc*, stand for the acceleration components.

Differentiate these equations with respect to time, to give:

$$\dot{a}_x = \alpha a_y + \beta a_z, \tag{30}$$

$$\dot{a}_y = -\alpha a_x + \gamma a_z, \tag{31}$$

$$\dot{a}_z = -\beta a_x - \gamma a_y. \tag{32}$$

Differentiating once more then produces

$$\ddot{a}_x = -\mu^2 a_x,$$

and similarly for a_y and a_z. This is a consequence of the fact that $\underline{q} = [-\gamma,\ \beta,\ -\alpha]^T$ is perpendicular to the acceleration vector (see Section 2.2).

Therefore, μ is observable if at least one of $a_x,\ a_y,\ a_z$ is non-zero†.

Now use equations (27) and (28) to formally express β and γ in terms of α:

$$\beta = \frac{1}{w}\left(a_x - \alpha v\right), \tag{33}$$

$$\gamma = \frac{1}{w}\left(a_y + \alpha u\right), \tag{34}$$

and substitute these into the equation for μ,

$$\mu^2 = \alpha^2 + \beta^2 + \gamma^2, \tag{35}$$

resulting in the following quadratic equation for α:

$$\alpha^2\left(u^2 + v^2 + w^2\right) + 2\alpha\left(ua_y - va_x\right) + \left(a_x^2 + a_y^2\right) - \mu^2 w^2 = 0. \tag{36}$$

Formally, the solutions to this then will be:

$$\alpha = \frac{1}{s^2}\left[-\left(ua_y - va_x\right) \pm \sqrt{\left(ua_y - va_x\right)^2 - s^2\left(a_x^2 + a_y^2 - \mu^2 w^2\right)}\right],$$

*In most situations only third-order derivatives are needed, but the analysis here avoids the numerous special cases that would thereby need to be considered.

†If all of the acceleration components are zero, then equations (30), (31) and (32) imply that all higher derivatives are also zero, indicating constant-velocity motion rather than a turning target.

where $s^2 = u^2 + v^2 + w^2$, thus providing two alternative numeric values for α.

Guidance as to which root should be chosen can be obtained by making use of equations (27), (28) and (29), so that the quantity under the square root is expressible in the form

$$w^2 (\gamma u - \beta v + \alpha w)^2 ,$$

giving

$$\alpha = \frac{1}{s^2} \Big[- (u a_y - v a_x) \pm w (\gamma u - \beta v + \alpha w) \Big],$$

and this is an identity only for the positive root. Therefore, α may be determined from the equation

$$\alpha = \frac{1}{s^2} \left[- (u a_y - v a_x) + \sqrt{(u a_y - v a_x)^2 - s^2 \left(a_x^2 + a_y^2 - \mu^2 w^2 \right)} \right],$$

which is valid unless $s = 0$.

Once α is available, β and γ can be obtained from equations (27) and (28) if $w \neq 0$, or from equations (30) and (31) if $w = 0$ and $a_z \neq 0$. If, however, both $w = 0$ and $a_z = 0$, the relevant available information consists of the two equations

$$\begin{aligned} 0 &= \beta u + \gamma v, \quad \text{from equation (29)}, \\ \mu^2 - \alpha^2 &= \beta^2 + \gamma^2, \quad \text{from equation (35)}. \end{aligned}$$

These can be solved provided $u^2 + v^2 \neq 0$ which — since $w = 0$ as well — also implies zero speed.

As an aside, equations (27) to (32) can be combined to yield the following:

$$\begin{aligned} \dot{a}_x &= -\mu^2 u - \gamma \, \underline{q}^T \underline{\dot{x}}, \\ \dot{a}_y &= -\mu^2 v + \beta \, \underline{q}^T \underline{\dot{x}}, \\ \dot{a}_z &= -\mu^2 w - \alpha \, \underline{q}^T \underline{\dot{x}}, \end{aligned}$$

where $\underline{\dot{x}} \equiv [u, v, w]^T$. Therefore, given μ and provided $\underline{q}^T \underline{\dot{x}} \neq 0$, α, β and γ may be determined straightforwardly. Note that the condition $\underline{q}^T \underline{\dot{x}} = 0$ implies motion in a circle (non-helical), since the velocity is then normal to the axial vector $\underline{q}$.

ABSTRACT

This paper defines a nonlinear kinematic model designed for tracking air vehicles undergoing helical-type motions, of which turning in the same plane forms a special case. For tracking purposes, a nine-state nonlinear Unscented Kalman Filter is demonstrated, using the process noise methods introduced in [4].

NIELINIOWY MODEL RUCHU DO ŚLEDZENIA CELÓW W TRAKCIE ZAWRACANIA

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W artykule określono nieliniowy model dynamiczny do śledzenia statków powietrznych podczas ruchu typu śruba, dla którego szczególnym przypadkiem jest zawracanie na płaszczyźnie. W celu zapewnienia zdolności śledzenia, przedstawiono 9-stanowy filtr UKF oparty na [4].

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