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PROBABILITY OF TRACK IMPACT IN DEFENDED AREA: USE OF GREEN'S THEOREM IN THE PLANE

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1. INTRODUCTION

When tracking a ballistic missile, whether of short or long range, it is frequently necessary to determine where the missile originated and where it is expected to land. This process requires extrapolating the track backwards or forwards in time and then characterising the resultant uncertainty domain on the earth's surface. The nature of the extrapolation and the coordinate conversions involved are outside the scope of the present paper, which concentrates instead on providing an answer to the question: does the predicted impact point fall within an area of interest? (here termed *defended area* for brevity).

It can be appreciated that a systematic answer needs to be provided in probabilistic terms, since the predicted track state and covariance between them define an uncertainty domain. Merely examining whether the predicted state (*i.e.* centroid) itself falls within the defended area is hardly satisfactory, and the same comment applies to using (say) the 95% track uncertainty ellipse (although a binary yes/no answer may be acceptable in some circumstances).

The track impact domain is usually assumed to be Gaussian in nature and this assumption is maintained here, although it is acknowledged that conversions from the native tracking coordinates into latitude and longitude (or east and north relative to some defined location) may result in a non-Gaussian distribution.

A straightforward and obvious way of extracting a probability of impact is to generate numerous random variates from within the impact distribution and determine the proportion of these variates that fall within the defended area, assumed closed. This method is simple to program but — in common with many Monte Carlo techniques — slow to converge. In fact, the uncertainty (variance) associated with the resulting probability is given by the equation (Appendix A):

$$\sigma_P^2 = \frac{P(1-P)}{N},$$

where P is the probability value and N is the number of variates; the rate of convergence is thus of the order of $1/\sqrt{N}$.

The present paper proposes an alternative approach, stemming from the recognition that what is sought is simply the quantity

$$P = \iint_A G(x,y)\,dxdy, \tag{1}$$

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where $G(x, y)$ stands for the bivariate probability density characterising the track state and uncertainty, and A is the area encompassed by the defended area, assumed closed. This quantity is what is estimated by the above Monte Carlo method, as a substitute for more formal (and typically complex) methods for evaluating a double integral.

An alternative method for calculating a double integral over some planar area is to invoke Green's theorem in the plane, which equates an area integral with a contour integral around its boundary. This approach is examined in Section 2.

A third alternative to the evaluation of equation (1) is direct numerical integration over the two-dimensional area A, and suitable techniques are provided in [5]. It can be appreciated, however, that for an area with a complex boundary, such an approach may well be complicated to program and potentially time-consuming in computational terms. Direct numerical integration is not, therefore, considered further here.

2. ANALYSIS

From [4], page 522, Green's theorem in the plane is of the form

$$\iint_A \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C \left(F_1 dx + F_2 dy \right),$$

where F_1 and F_2 are functions of x and y, A is a finite area and the contour C is the boundary of that area, traced anti-clockwise*.

Now suppose that the left-hand integral needs to be over the function

$$G(x, y) = \frac{1}{2\pi\sigma_x\sigma_y} \exp\left\{ -\frac{1}{2} \left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} \right) \right\},$$

standing for the bivariate normal distribution, in coordinates centred at the track impact state (*i.e.* centroid) and aligned with the principal axes of its covariance. The transformation from more general two-dimensional track-based coordinates, whether east-north or longitude-latitude, is discussed in Appendix B.

Set $F_1 = 0$ and F_2 such that

$$F_2 = \int^x G(x, y) dx.$$

Therefore,

$$P = \iint_A G(x, y) dxdy = \oint_C F_2 \, dy, \tag{2}$$

using the definition from equation (1).

Repeat the process with $F_2 = 0$ and

$$F_1 = -\int^y G(x, y) dy,$$

so that

$$P = \iint_A G(x, y) dxdy = \oint_C F_1 \, dx. \tag{3}$$

*That is, the region A is on the left when advancing in the direction of contour integration.

Add equations (2) and (3) to give

$$P = \iint_A G(x,y)dxdy = \frac{1}{2}\oint_C (F_2\, dy + F_1\, dx)\,. \tag{4}$$

Now, from the above definitions of F_2 and F_1:

$$F_2 = \frac{1}{2\pi\sigma_x\sigma_y} e^{-y^2/2\sigma_y^2} \int^x e^{-x^2/2\sigma_x^2}\, dx,$$
$$F_1 = -\frac{1}{2\pi\sigma_x\sigma_y} e^{-x^2/2\sigma_x^2} \int^y e^{-y^2/2\sigma_y^2}\, dy.$$

Therefore, the required area integral becomes:

$$P = \frac{1}{4\pi\sigma_x\sigma_y}\oint_C \left[e^{-y^2/2\sigma_y^2}\left\{\int^x e^{-x^2/2\sigma_x^2}\, dx\right\} dy - e^{-x^2/2\sigma_x^2}\left\{\int^y e^{-y^2/2\sigma_y^2}\, dy\right\} dx\right].$$

Both internal integrals are of the same form and may be written

$$\int^\xi e^{-\tau^2/2\alpha^2} d\tau = \alpha\sqrt{2}\int^{\xi/\alpha\sqrt{2}} e^{-\theta^2} d\theta,$$
$$\equiv \alpha\sqrt{2}\, N\left(\xi/\alpha\sqrt{2}\right),$$

after a change of integration variable and defining the function $N(\mu)$ as follows:

$$N(\mu) = \int^\mu e^{-\theta^2} d\theta.$$

Therefore,

$$P = \frac{\sqrt{2}}{4\pi\sigma_x\sigma_y}\oint_C \left[\sigma_x e^{-y^2/2\sigma_y^2} N\left(x/\sigma_x\sqrt{2}\right) dy - \sigma_y e^{-x^2/2\sigma_x^2} N\left(y/\sigma_y\sqrt{2}\right) dx\right]. \tag{5}$$

Now suppose that the continuous closed contour C can be expressed in polygonal form and consider one linear section from (a_1, b_1) to (a_2, b_2), with associated contribution P_i. Change to parametric coordinates over this line interval, such that

$$x = a_1 + t\,(a_2 - a_1)\,,$$
$$y = b_1 + t\,(b_2 - b_1)\,,$$

and where $t \in [0,1]$.

Therefore,

$$P_i = \frac{\sqrt{2}}{4\pi\sigma_x\sigma_y}\int_{t=0}^1 \left[\sigma_x\,(b_2 - b_1)\, e^{-y^2(t)/2\sigma_y^2} N\left(x(t)/\sigma_x\sqrt{2}\right)\right.$$
$$\left. -\ \sigma_y\,(a_2 - a_1)\, e^{-x^2(t)/2\sigma_x^2} N\left(y(t)/\sigma_y\sqrt{2}\right)\right] dt,$$
$$= \frac{\sqrt{2}}{4\pi\sigma_x\sigma_y}\left[\sigma_x\,(b_2 - b_1)\int_{t=0}^1 e^{-y^2(t)/2\sigma_y^2} N\left(x(t)/\sigma_x\sqrt{2}\right) dt\right.$$
$$\left. -\ \sigma_y\,(a_2 - a_1)\int_{t=0}^1 e^{-x^2(t)/2\sigma_x^2} N\left(y(t)/\sigma_y\sqrt{2}\right) dt\right].$$

To proceed further, it is necessary to obtain an analytic expression for $N(\mu)$, which is readily supplied by equation 2.33(1) on page 108 of reference [2], namely:

$$\int e^{-\left(ax^2+2bx+c\right)}dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{b^2-ac}{a}\right)\text{ERF}\left(x\sqrt{a}+\frac{b}{\sqrt{a}}\right), \quad a \neq 0,$$

where ERF(.) is the error function*. In the present case, $b = c = 0$ and $a = 1$, so

$$N(\mu) = \frac{\sqrt{\pi}}{2}\text{ERF}(\mu).$$

Substituting this into the above expression for P_i:

$$P_i = \frac{1}{4\sqrt{2\pi}\sigma_x\sigma_y}\left[\sigma_x\left(b_2-b_1\right)\int_{t=0}^{1} e^{-y^2(t)/2\sigma_y^2}\,\text{ERF}\left(x(t)/\sigma_x\sqrt{2}\right)dt - \sigma_y\left(a_2-a_1\right)\int_{t=0}^{1} e^{-x^2(t)/2\sigma_x^2}\,\text{ERF}\left(y(t)/\sigma_y\sqrt{2}\right)dt\right]. \tag{6}$$

This sectional integral may be evaluated numerically and the total integral P then obtained as

$$P = \sum_i P_i, \tag{7}$$

summing over the boundary line segments in anti-clockwise order.

The example below compares the results of using equation (7) with the Monte Carlo method, for a simple five-sided polygonal shape. An example bounded area is shown in Figure 1, in conjunction with a representative Gaussian track probability domain, having $\sigma_x = 0.7$, $\sigma_y = 0.5$ in arbitrary units.

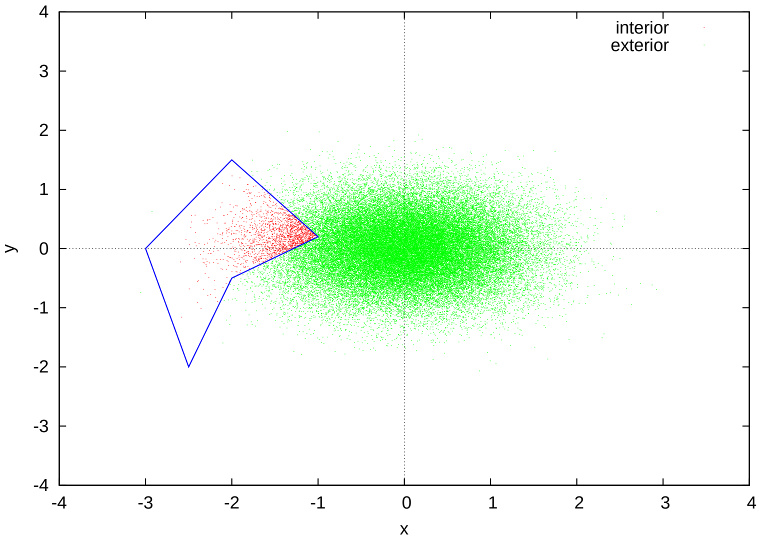

Fig. 1. Example of defended area and track probability distribution

*Provided in C by the math function of the same name.

Here, 60,000 sample variates have been generated, with each such variate colour-coded depending on whether it is inside or outside the defended (blue-bounded) area*. Note that, for convenience, the origin of the coordinate system has here been chosen at the track centroid and with axes aligned with the principal axes of the track covariance.

From an inspection of Figure 1, it can be anticipated that the actual impact probability is likely to be small, since most of the track distribution falls outside the defended area. In fact, the Monte Carlo method estimates that $P \approx 0.0272$, with associated uncertainty 6.6×10^{-4} (see Section 1). Applying the boundary integration method of the present section gives $P = 0.027227$, which is gratifyingly close.

A somewhat more definitive comparison can be made if the Monte Carlo analysis is repeated over a much longer total sample size (here 6 million), but with intermediate probabilities and associated uncertainties calculated at every increase of 40,000. The results are shown in Figure 2, with the error bars reflecting a one-sigma uncertainty in the associated probability estimate.

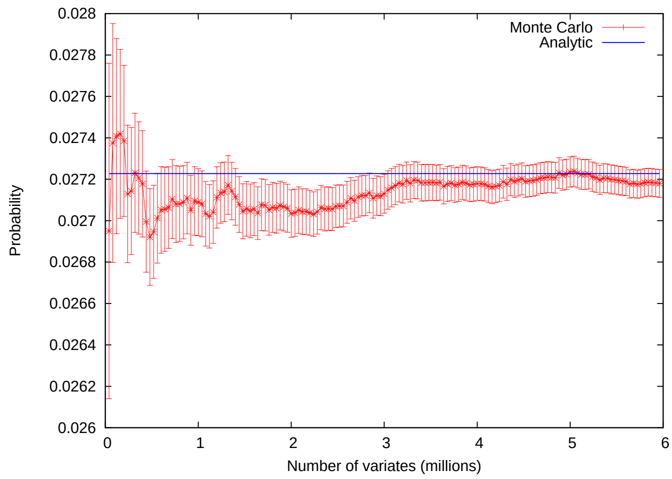

Fig. 2. Comparison of Monte Carlo method with equation (7)

It can be seen that the probability estimates for the two methods are comparable, although the slow convergence of the Monte Carlo method is also evident.

3. CONCLUSIONS

This paper provides a method for determining the probability that the impact point of a track falls within a defended area, by integrating a function around the boundary, here assumed of polygonal form. This method is more complex to program but offers improved computational efficiency compared to the more straightforward Monte Carlo technique, which counts the proportion of random variates falling within the area.

*Several efficient numerical techniques are available on the internet to determine whether or not a point falls within a two-dimensional closed polygon.

A defended area defined in terms of curved sections can be accommodated by means of a polygonal approximation, or else the integral given by equation (5) can be numerically evaluated to any desired accuracy.

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A. DISCRETE BINARY RANDOM SEQUENCE

This section determines the uncertainty associated with the impact probability P, based on the following operational sequence:

1. Initiate a counter C to zero.
2. Generate a random sample vector $\underline{z}_i$ from within the distribution.
3. If such a vector $\underline{z}_i$ falls within the defended area, increment the counter C by unity.
4. Once N samples have been generated, determine the ratio $P = C/N$.

Examining the above sequence from a Kalman Filter perspective ([1]), it is evident that a 'measurement' is provided at step 3, and this will take the form of zero (if $\underline{z}_i$ falls outside the defended area) or unity (if inside). Therefore, if the uncertainty σ_m associated with this single-sample measurement can be determined, the uncertainty associated with P follows directly:

$$\sigma_P = \frac{\sigma_m}{\sqrt{N}}. \tag{8}$$

It remains to determine σ_m and it turns out that its value is related to P itself.

The logic can be formalised by introducing the quantity ζ_i, which is the individual result of each comparison of the sample track with the defended area. The quantity ζ_i will take either a value of zero, when the sample is outside the defended area, or unity if it is inside. The result of a large number of such samples is then a *binary sequence* — a stream of ones or zeros — and ζ_i is a Bernoulli random variable ([6], chapter 2 or [3], chapter 3). Since it is the proportion of samples falling inside the defended area that is of interest, by definition the expected value of ζ_i will be P. That is,

$$E\{\zeta_i\} \equiv \bar{\zeta} = P.$$

The other main quantity that is required is the standard deviation σ_m associated with ζ_i, which is given by

$$\sigma_m^2 = \frac{1}{N}\sum(\zeta_i - P)^2 .$$

If this is expanded out,

$$\sigma_m^2 = P^2 + \frac{1}{N}\sum\left(\zeta_i^2 - 2P\zeta_i\right) . \tag{9}$$

Now suppose that out of the N samples, there are M_i that fall within the defended area and M_o that fall outside. Therefore, $P = M_i/N$.

Also, since ζ_i takes only binary values,

$$\sum \zeta_i^2 = M_i \quad \text{and} \quad \sum \zeta_i = M_i.$$

Substituting these into equation (9):

$$\begin{aligned} \sigma_m^2 &= P^2 + \frac{M_i}{N} - 2P\frac{M_i}{N}, \\ &= P^2 + P - 2P^2, \\ &= P(1-P), \end{aligned}$$

which is the result sought.

B. TRANSFORMATION TO PRINCIPAL AXES

It is assumed that the track state and covariance have already been mapped to some convenient earth-surface-based, two-dimensional planar coordinates, here defined in terms of the vector $\underline{z} = [\xi,\, \eta]^T$. Thus, the probability density characterising the track state and uncertainty is assumed to be of the following form:

$$G(\xi,\eta) = \frac{1}{2\pi\sqrt{|\boldsymbol{P}_z|}} \exp\left\{-\frac{1}{2}\left(\underline{z}-\underline{z}_0\right)^T \boldsymbol{P}_z^{-1}\left(\underline{z}-\underline{z}_0\right)\right\},$$

where $\underline{z}_0 = [\xi_0,\, \eta_0]^T$ stands for the track centroid and $\boldsymbol{P}_z$ for the associated covariance matrix.

The aim is to define a coordinate shift and rotation from $\xi,\, \eta$ to $x,\, y$ such that the transformed matrix $\boldsymbol{P'}$ is of diagonal form. That is, set

$$\underline{z} - \underline{z}_0 = \boldsymbol{R}\underline{x}, \;\; \Rightarrow \underline{x} = \boldsymbol{R}^T\left(\underline{z}-\underline{z}_0\right), \tag{10}$$

where $\underline{x} = (x,\, y)$ and $\boldsymbol{R}$ is a rotation matrix defined as

$$\boldsymbol{R} = \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix}$$

for some rotation angle ψ.

Therefore, require

$$\boldsymbol{Q}^{-1} \equiv (\boldsymbol{R}\underline{x})^T \boldsymbol{P}_z^{-1}(\boldsymbol{R}\underline{x})$$

to be in diagonal form.

Given the properties of rotation matrices, and combined with the equality $(\boldsymbol{AC})^{-1} = \boldsymbol{C}^{-1}\boldsymbol{A}^{-1}$ for any two matrices $\boldsymbol{A}$ and $\boldsymbol{C}$, it is not difficult to show that

$$\boldsymbol{Q}^{-1} = \left(\boldsymbol{R}^T \boldsymbol{P}_z \boldsymbol{R}\right)^{-1},$$

so that it is sufficient to define $\boldsymbol{R}$ such that the product $\boldsymbol{R}^T\boldsymbol{P}_z\boldsymbol{R}$ is in diagonal form. That is, require

$$\begin{bmatrix} Q_{xx} & 0 \\ 0 & Q_{yy} \end{bmatrix} = \begin{bmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{bmatrix} \begin{bmatrix} P_{\xi\xi} & P_{\xi\eta} \\ P_{\xi\eta} & P_{\eta\eta} \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{bmatrix},$$

$$= \begin{bmatrix} \left(P_{\xi\xi}\cos^2\psi - P_{\xi\eta}\sin 2\psi + P_{\eta\eta}\sin^2\psi\right) & \left(P_{\xi\eta}\cos 2\psi - (P_{\eta\eta} - P_{\xi\xi})\sin\psi\cos\psi\right) \\ \left(P_{\xi\eta}\cos 2\psi - (P_{\eta\eta} - P_{\xi\xi})\sin\psi\cos\psi\right) & \left(P_{\xi\xi}\sin^2\psi + P_{\xi\eta}\sin 2\psi + P_{\eta\eta}\cos^2\psi\right) \end{bmatrix}.$$

The rotation angle ψ is then obtained by setting

$$P_{\xi\eta}\cos 2\psi - (P_{\eta\eta} - P_{\xi\xi})\sin\psi\cos\psi = 0,$$

which implies that

$$\tan 2\psi = \frac{2P_{\xi\eta}}{P_{\eta\eta} - P_{\xi\xi}}.$$

With this value of ψ, the diagonal elements of the covariance matrix $\boldsymbol{Q}$ in the transformed coordinates are given by

$$Q_{xx} = P_{\xi\xi}\cos^2\psi - P_{\xi\eta}\sin 2\psi + P_{\eta\eta}\sin^2\psi,$$
$$Q_{yy} = P_{\xi\xi}\sin^2\psi + P_{\xi\eta}\sin 2\psi + P_{\eta\eta}\cos^2\psi.$$

In the present context, it is immaterial which is the ellipse semi-major and which the semi-minor axis.

Note that in order to make use of equation (6), it is necessary to map each vertex on the polygon from ξ, η coordinates to x, y using equation (10).

ABSTRACT

This paper discusses a simple method for determining the probability that the predicted impact point of a track falls within a defended area, by integrating a function around the boundary of that area. The proposed method is compared to a more direct but computationally intensive Monte Carlo technique.

OKREŚLANIE PRAWDOPODOBIEŃSTWA ZAKOŃCZENIA TRAJEKTORII W CHRONIONYM OBSZARZE: ZASTOSOWANIE TWIERDZENIA GREENA NA PLASZCZYŹNIE

Paul F. Easthope

W artykule przedyskutowano prostą metodę określania prawdopodobieństwa przynależności przewidywanego punktu zakończenia trajektorii do wnętrza obszaru chronionego poprzez całkowanie pewnej funkcji w pobliżu granicy tego obszaru. Zaproponowana metoda jest porównana z bardziej bezpośrednią, lecz bardziej wymagająca obliczeniowo metoda Monte Carlo.

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