

Open mathematical problems which cannot be stated formally as they refer to intuitive meanings of mathematical formulae and the current mathematical knowledge

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Abstract

Let $\beta = ((24!)!)!$, and let \mathcal{P}_{n^2+1} denote the set of all primes of the form $n^2 + 1$. Let \mathcal{M} denote the set of all positive multiples of elements of the set $\mathcal{P}_{n^2+1} \cap (\beta, \infty)$. The set $\mathcal{X} = \{0, \dots, \beta\} \cup \mathcal{M}$ satisfies the following conditions: (1) $\text{card}(\mathcal{X})$ is greater than a huge positive integer and it is conjectured that \mathcal{X} is infinite, (2) we do not know any algorithm deciding the finiteness of \mathcal{X} , (3) a known and short algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in \mathcal{X}$, (4) a known and short algorithm returns an integer n such that \mathcal{X} is infinite if and only if \mathcal{X} contains an element greater than n . The following problem is open: define a set $\mathcal{X} \subseteq \mathbb{N}$ such that \mathcal{X} satisfies conditions (1)-(4) and a known and simple formula $\varphi(x)$ satisfies $\mathcal{X} = \{n \in \mathbb{N} : \varphi(n)\}$, where $\varphi(n)$ has the same intuitive meaning for every $n \in \mathbb{N}$ (5). The statements $\varphi(n)$ in condition (5) have always the same intuitive meaning, if the predicate $\varphi(x)$ expresses a *natural property*, the term propounded by the philosopher David Lewis (1941-2001). Let $f(3) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geq 3$. For an integer $n \geq 3$, let Ψ_n denote the following statement: *if a system of equations $\mathcal{S} \subseteq \{x_i! = x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i \cdot x_j = x_{j+1} : 1 \leq i \leq j \leq n-1\}$ has only finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq f(n)$* . We prove that for every statement Ψ_n the bound $f(n)$ cannot be decreased. The author's guess is that the statements Ψ_3, \dots, Ψ_9 are true. We prove that the statement Ψ_9 implies that the set \mathcal{X} of all non-negative integers n whose number of digits belongs to \mathcal{P}_{n^2+1} satisfies conditions (1)-(5).

Key words and phrases: computable set $\mathcal{X} \subseteq \mathbb{N}$ whose finiteness remains conjectured, computable set $\mathcal{X} \subseteq \mathbb{N}$ whose infiniteness remains conjectured, David Lewis's notion of a natural property, huge integers for which arithmetical operations cannot be performed by any physical process, intuitive meaning of a mathematical formula, Zenkin's super-induction.

1 Introduction and basic definitions and lemmas

In this article, we discuss open problems on computable sets $\mathcal{X} = \{n \in \mathbb{N} : \varphi(n)\}$ which cannot be stated formally as they require that the finiteness (infiniteness) of \mathcal{X} remains conjectured and $\varphi(n)$ has the same intuitive meaning for every $n \in \mathbb{N}$.

Definition 1. Let $\beta = ((24!)!)!$.

Lemma 1. $\beta \approx 10^{10^{25.16114896940657}}$.

Proof. We ask Wolfram Alpha at <http://wolframalpha.com>. □

Definition 2. We say that an integer $m \geq -1$ is a *threshold number* of a set $\mathcal{X} \subseteq \mathbb{N}$, if \mathcal{X} is infinite if and only if \mathcal{X} contains an element greater than m , cf. [11] and [12].

If a set $\mathcal{X} \subseteq \mathbb{N}$ is empty or infinite, then any integer $m \geq -1$ is a threshold number of \mathcal{X} . If a set $\mathcal{X} \subseteq \mathbb{N}$ is non-empty and finite, then the all threshold numbers of \mathcal{X} form the set $\{\max(\mathcal{X}), \max(\mathcal{X}) + 1, \max(\mathcal{X}) + 2, \dots\}$.

Definition 3. We say that a non-negative integer m is a weak threshold number of a set $X \subseteq \mathbb{N}$, if X is infinite if and only if $\text{card}(X) > m$.

Theorem 1. For every $X \subseteq \mathbb{N}$, if an integer $m \geq -1$ is a threshold number of X , then $m + 1$ is a weak threshold number of X .

Proof. For every $X \subseteq \mathbb{N}$, if $m \in [-1, \infty) \cap \mathbb{Z}$ and $\text{card}(X) > m + 1$, then $X \cap [m + 1, \infty) \neq \emptyset$. □

We do not know any weak threshold number of the set of all primes of the form $n^2 + 1$. The same is true for the sets

$$\left\{ n \in \mathbb{N} : 2^{2^n} + 1 \text{ is composite} \right\}$$

and

$$\{ n \in \mathbb{N} : n! + 1 \text{ is a square} \}$$

Lemma 2. For every positive integers x and y , $x! \cdot y = y!$ if and only if

$$(x + 1 = y) \vee (x = y = 1)$$

Lemma 3. (Wilson's theorem, [2, p. 89]). For every integer $x \geq 2$, x is prime if and only if x divides $(x - 1)! + 1$.

2 Open Problems 1 and 2

The following two open problems cannot be stated formally as they refer to intuitive meanings of mathematical formulae and the current mathematical knowledge.

Open Problem 1. Define a set $X \subseteq \mathbb{N}$ that satisfies the following conditions:

- (1) $\text{card}(X)$ is greater than a huge positive integer and it is conjectured that X is infinite,
- (2) we do not know any algorithm deciding the finiteness of X ,
- (3) a known and short algorithm for every $n \in \mathbb{N}$ decides whether or not $n \in X$,
- (4•) a known and short algorithm returns an integer n such that X is infinite if and only if $\text{card}(X) > n$,
- (5) a known and simple formula $\varphi(x)$ satisfies $X = \{n \in \mathbb{N} : \varphi(n)\}$, where $\varphi(n)$ has the same intuitive meaning for every $n \in \mathbb{N}$.

Open Problem 2. Define a set $X \subseteq \mathbb{N}$ such that X satisfies conditions (1)–(3), (5), and a known and short algorithm returns an integer n such that X is infinite if and only if X contains an element greater than n (4).

The statements $\varphi(n)$ in condition (5) have always the same intuitive meaning, if the predicate $\varphi(x)$ expresses David Lewis's *natural property*. For the meaning of this term, the reader is referred to [1].

Theorem 2. Open Problem 2 claims more than Open Problem 1.

Proof. By Theorem 1, condition (4) implies condition (4•). □

3 Two partial solutions to Open Problem 2

Edmund Landau's conjecture states that the set \mathcal{P}_{n^2+1} of all primes of the form $n^2 + 1$ is infinite, see [5, pp. 37–38] and [8]. Let \mathcal{M} denote the set of all positive multiples of elements of the set $\mathcal{P}_{n^2+1} \cap (\beta, \infty)$.

Theorem 3. The set $X = \{0, \dots, \beta\} \cup \mathcal{M}$ satisfies conditions (1)–(4).

Proof. Condition (1) holds as $\text{card}(X) > \beta$ and the set \mathcal{P}_{n^2+1} is conjecturally infinite. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of \mathcal{P}_{n^2+1} is greater than β . Thus condition (2) holds. Condition (3) holds trivially. Since the set \mathcal{M} is empty or infinite, the integer β is a threshold number of X . Thus condition (4) holds. □

Let $[\cdot]$ denote the integer part function.

Lemma 4. For every non-negative integer n , $\left[\frac{3n-3\beta+3}{3n-3\beta+2}\right]$ equals 0 or 1. The first case holds when $n \leq \beta - 1$. The second case holds when $n \geq \beta$.

Lemma 5. The function

$$\mathbb{N} \cap [\beta, \infty) \ni n \xrightarrow{\theta} \beta + n - [\sqrt{n}]^2 \in \mathbb{N} \cap [\beta, \infty)$$

takes every integer value $k \geq \beta$ infinitely many times.

Proof. Let $t = k - \beta$. The equality $\theta(n) = k$ holds for every

$$n \in \{(t+0)^2 + t, (t+1)^2 + t, (t+2)^2 + t, \dots\} \cap [\beta, \infty)$$

□

Theorem 4. The set $\mathcal{X} = \left\{n \in \mathbb{N} : 2 + \left[\frac{3n-3\beta+3}{3n-3\beta+2}\right] \cdot \left(\left(\beta + n - [\sqrt{n}]^2\right)^2 - 1\right) \text{ is prime}\right\}$ satisfies conditions (1)-(4).

Proof. Condition (3) holds trivially. By Lemma 4, $\mathcal{X} = \{0, \dots, \beta - 1\} \cup \mathcal{H}$, where

$$\mathcal{H} = \left\{n \in \mathbb{N} \cap [\beta, \infty) : \left(\beta + n - [\sqrt{n}]^2\right)^2 + 1 \text{ is prime}\right\}$$

By Lemma 5, the set \mathcal{H} is empty or infinite. The second case holds when

$$\exists k \in \mathbb{N} \cap [\beta, \infty) \quad k^2 + 1 \text{ is prime} \quad (6)$$

The equality $\mathcal{X} = \{0, \dots, \beta - 1\} \cup \mathcal{H}$ and the last two sentences imply that $\beta - 1$ is a threshold number of \mathcal{X} and conditions (1) and (4) hold. Condition (2) holds as due to known physics we are not able to confirm statement (6) by a direct computation. □

4 The statements Ψ_n which seem to be true for every $n \in \{3, \dots, 9\}$

Let $f(3) = 4$, and let $f(n+1) = f(n)!$ for every integer $n \geq 3$. For an integer $n \geq 3$, let \mathcal{U}_n denote the following system of equations:

$$\begin{cases} \forall i \in \{1, \dots, n-1\} \setminus \{2\} \quad x_i! = x_{i+1} \\ x_1 \cdot x_2 = x_3 \\ x_2 \cdot x_2 = x_3 \end{cases}$$

The diagram in Figure 1 illustrates the construction of the system \mathcal{U}_n .

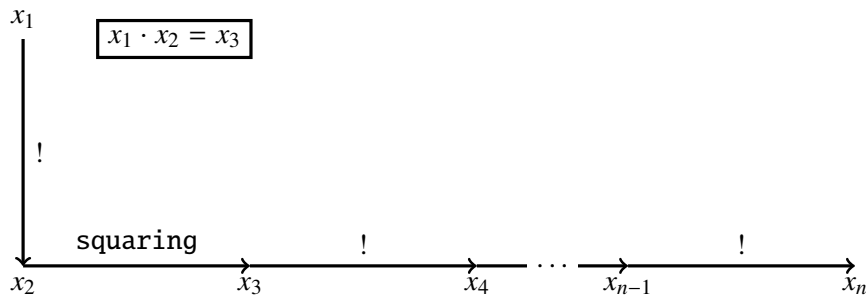


Fig. 1 Construction of the system \mathcal{U}_n

Lemma 6. For every integer $n \geq 3$, the system \mathcal{U}_n has exactly two solutions in positive integers, namely $(1, \dots, 1)$ and $(2, 2, f(3), \dots, f(n))$.

Let

$$B_n = \{x_i! = x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_i \cdot x_j = x_{j+1} : 1 \leq i \leq j \leq n-1\}$$

For an integer $n \geq 3$, let Ψ_n denote the following statement: if a system of equations $\mathcal{S} \subseteq B_n$ has only finitely many solutions in positive integers x_1, \dots, x_n , then each such solution (x_1, \dots, x_n) satisfies $x_1, \dots, x_n \leq f(n)$. The statement Ψ_n says that for subsystems of B_n with a finite number of solutions, the largest known solution is indeed the largest possible. The author's guess is that the statements Ψ_3, \dots, Ψ_9 are true.

Theorem 5. Every statement Ψ_n is true with an unknown integer bound that depends on n .

Proof. For every positive integer n , the system B_n has a finite number of subsystems. □

Theorem 6. For every statement Ψ_n , the bound $f(n)$ cannot be decreased.

Proof. It follows from Lemma 6 because $\mathcal{U}_n \subseteq B_n$. □

5 The statement Ψ_9 solves Open Problem 2

Let \mathcal{A} denote the following system of equations:

$$\left\{ \begin{array}{l} x_2! = x_3 \\ x_3! = x_4 \\ x_5! = x_6 \\ x_8! = x_9 \\ x_1 \cdot x_1 = x_2 \\ x_3 \cdot x_5 = x_6 \\ x_4 \cdot x_8 = x_9 \\ x_5 \cdot x_7 = x_8 \end{array} \right.$$

Lemma 2 and the diagram in Figure 2 explain the construction of the system \mathcal{A} .

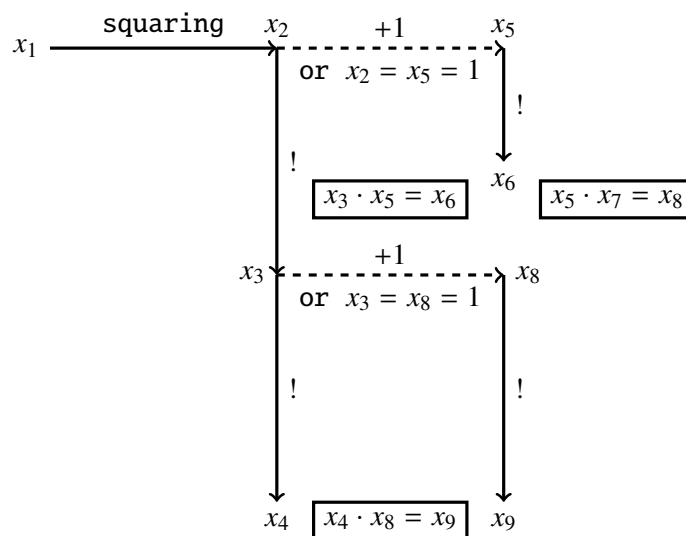


Fig. 2 Construction of the system \mathcal{A}

Lemma 7. For every integer $x_1 \geq 2$, the system \mathcal{A} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ is prime. In this case, the integers x_2, \dots, x_9 are uniquely determined by the following equalities:

$$\begin{aligned} x_2 &= x_1^2 \\ x_3 &= (x_1^2)! \\ x_4 &= ((x_1^2)!)! \\ x_5 &= x_1^2 + 1 \\ x_6 &= (x_1^2 + 1)! \\ x_7 &= \frac{(x_1^2)! + 1}{x_1^2 + 1} \\ x_8 &= (x_1^2)! + 1 \\ x_9 &= ((x_1^2)! + 1)! \end{aligned}$$

Proof. By Lemma 2, for every integer $x_1 \geq 2$, the system \mathcal{A} is solvable in positive integers x_2, \dots, x_9 if and only if $x_1^2 + 1$ divides $(x_1^2)! + 1$. Hence, the claim of Lemma 7 follows from Lemma 3. \square

Lemma 8. There are only finitely many tuples $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ which solve the system \mathcal{A} and satisfy $x_1 = 1$.

Proof. If a tuple $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$ solves the system \mathcal{A} and $x_1 = 1$, then $x_1, \dots, x_9 \leq 2$. Indeed, $x_1 = 1$ implies that $x_2 = x_1^2 = 1$. Hence, for example, $x_3 = x_2! = 1$. Therefore, $x_8 = x_3 + 1 = 2$ or $x_8 = 1$. Consequently, $x_9 = x_8! \leq 2$. \square

Theorem 7. The statement Ψ_9 proves the following implication: if there exists an integer $x_1 \geq 2$ such that $x_1^2 + 1$ is prime and greater than $f(7)$, then the set \mathcal{P}_{n^2+1} is infinite.

Proof. Suppose that the antecedent holds. By Lemma 7, there exists a unique tuple $(x_2, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^8$ such that the tuple (x_1, x_2, \dots, x_9) solves the system \mathcal{A} . Since $x_1^2 + 1 > f(7)$, we obtain that $x_1^2 \geq f(7)$. Hence, $(x_1^2)! \geq f(7)! = f(8)$. Consequently,

$$x_9 = ((x_1^2)! + 1)! \geq (f(8) + 1)! > f(8)! = f(9)$$

Since $\mathcal{A} \subseteq B_9$, the statement Ψ_9 and the inequality $x_9 > f(9)$ imply that the system \mathcal{A} has infinitely many solutions $(x_1, \dots, x_9) \in (\mathbb{N} \setminus \{0\})^9$. According to Lemmas 7 and 8 the set \mathcal{P}_{n^2+1} is infinite. \square

Let \mathcal{F} denote the set of all non-negative integers k whose number of digits belongs to \mathcal{P}_{n^2+1} .

Lemma 9. $\text{card}(\mathcal{F}) \geq 9 \cdot 10^9 \cdot 4^{747}$.

Proof. The following PARI/GP ([7]) command

```
(12:26) gp > isprime(1+9*4^747, {flag=2})
%1 = 1
```

is shown together with its output. This command performs the APRCL primality test, the best deterministic primality test algorithm ([10, p. 226]). It rigorously shows that the number $(3 \cdot 2^{747})^2 + 1$ is prime. Since $9 \cdot 10^9 \cdot 4^{747}$ non-negative integers have $1 + 9 \cdot 4^{747}$ digits, the desired inequality holds. \square

Theorem 8. The statement Ψ_9 implies that $X = \mathcal{F}$ satisfies conditions (1)–(5).

Proof. Suppose that the antecedent holds. Since the set \mathcal{P}_{n^2+1} is conjecturally infinite, Lemma 9 implies condition (1). Conditions (3) and (5) hold trivially. By Theorem 7, $\underbrace{9 \dots 9}_{\beta \text{ digits}}$ is a threshold number of X . Thus condition (4) holds. By Lemma 1, due to known physics we are not able to confirm by a direct computation that some element of \mathcal{P}_{n^2+1} is greater than $f(7) = ((24!)!)! = \beta$. Thus condition (2) holds. \square

6 Open Problems 3 and 4

Definition 4. Let $(1\diamond)$ denote the following condition: $\text{card}(X)$ is greater than a huge positive integer and it is conjectured that $X = \mathbb{N}$.

Definition 5. Let $(2\diamond)$ denote the following condition: we do not know any algorithm deciding the equality $X = \mathbb{N}$.

The following two open problems cannot be stated formally as they refer to intuitive meanings of mathematical formulae and the current mathematical knowledge.

Open Problem 3. Define a set $X \subseteq \mathbb{N}$ that satisfies conditions $(1\diamond)$ - $(2\diamond)$, (2) - (3) , $(4\bullet)$, and (5) .

Open Problem 3 claims more than Open Problem 1 as condition $(1\diamond)$ implies condition (1) .

Open Problem 4. Define a set $X \subseteq \mathbb{N}$ that satisfies conditions $(1\diamond)$ - $(2\diamond)$ and (2) - (5) .

Open Problem 4 claims more than Open Problem 2 as condition $(1\diamond)$ implies condition (1) .

Theorem 9. Open Problem 4 claims more than Open Problem 3.

Proof. By Theorem 1, condition (4) implies condition $(4\bullet)$. □

7 A partial solution to Open Problem 4

Let \mathcal{V} denote the set of all positive multiples of elements of the set

$$\{n \in \{\beta + 1, \beta + 2, \beta + 3, \dots\} : 2^{2^n} + 1 \text{ is composite}\}$$

Theorem 10. The set $X = \{0, \dots, \beta\} \cup \mathcal{V}$ satisfies conditions $(1\diamond)$ - $(2\diamond)$ and (2) - (4) .

Proof. The inequality $\text{card}(X) > \beta$ holds trivially. Most mathematicians believe that $2^{2^n} + 1$ is composite for every integer $n \geq 5$, see [3, p. 23]. These two facts imply conditions $(1\diamond)$ and $(2\diamond)$. Condition (3) holds trivially. Since the set \mathcal{V} is empty or infinite, the integer β is a threshold number of X . Thus condition (4) holds. The question of finiteness of the set $\{n \in \mathbb{N} : 2^{2^n} + 1 \text{ is composite}\}$ remains open, see [4, p. 159]. Hence, the question of emptiness of the set

$$\{n \in \{\beta + 1, \beta + 2, \beta + 3, \dots\} : 2^{2^n} + 1 \text{ is composite}\}$$

remains open. Therefore, the question of finiteness of the set \mathcal{V} remains open. Consequently, the question of finiteness of the set X remains open and condition (2) holds. □

8 Open Problems 5 and 6

Definition 6. Let (1^*) denote the following condition: $\text{card}(X)$ is greater than a huge positive integer and it is conjectured that X is finite.

The following two open problems cannot be stated formally as they refer to intuitive meanings of mathematical formulae and the current mathematical knowledge.

Open Problem 5. Define a set $X \subseteq \mathbb{N}$ that satisfies conditions (1^*) , (2) - (3) , $(4\bullet)$, and (5) .

Open Problem 6. Define a set $X \subseteq \mathbb{N}$ that satisfies conditions (1^*) and (2) - (5) .

Theorem 11. Open Problem 6 claims more than Open Problem 5.

Proof. By Theorem 1, condition (4) implies condition $(4\bullet)$. □

9 A partial solution to Open Problem 6

A weak form of Szpiro's conjecture implies that there are only finitely many solutions to the equation $x! + 1 = y^2$, see [6].

Lemma 10. ([9, p. 297]). *It is conjectured that $x! + 1$ is a square only for $x \in \{4, 5, 7\}$.*

Let \mathcal{W} denote the set of all integers x greater than β such that $x! + 1$ is a square.

Theorem 12. *The set*

$$\mathcal{X} = \{0, \dots, \beta\} \cup \{k \cdot x : (k \in \mathbb{N} \setminus \{0\}) \wedge (x \in \mathcal{W})\}$$

satisfies conditions (1) and (2)-(4).*

Proof. Condition (1*) holds as $\text{card}(\mathcal{X}) > \beta$ and the set \mathcal{W} is conjecturally empty by Lemma 10. Condition (3) holds trivially. We do not know any algorithm that decides the emptiness of \mathcal{W} and the set

$$\mathcal{Y} = \{k \cdot x : (k \in \mathbb{N} \setminus \{0\}) \wedge (x \in \mathcal{W})\}$$

is empty or infinite. Thus condition (2) holds. Since the set \mathcal{Y} is empty or infinite, the integer β is a threshold number of \mathcal{X} . Thus condition (4) holds. \square

10 The statement Ψ_6 solves Open Problem 6

Let C denote the following system of equations:

$$\begin{cases} x_1! = x_2 \\ x_2! = x_3 \\ x_5! = x_6 \\ x_4 \cdot x_4 = x_5 \\ x_3 \cdot x_5 = x_6 \end{cases}$$

Lemma 2 and the diagram in Figure 3 explain the construction of the system C .

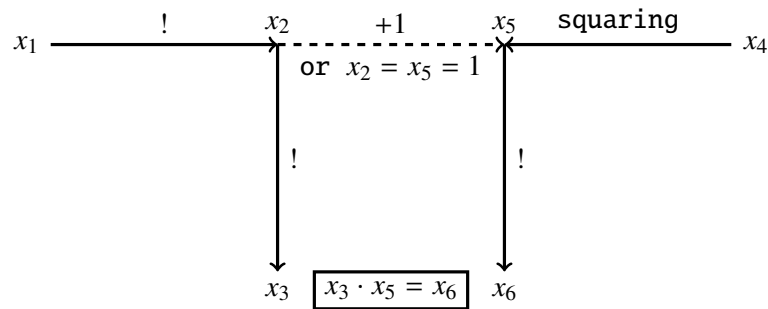


Fig. 3 Construction of the system C

Lemma 11. *For every $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$, the system C is solvable in positive integers x_2, x_3, x_5, x_6 if and only if $x_1! + 1 = x_4^2$. In this case, the integers x_2, x_3, x_5, x_6 are uniquely determined by the following equalities:*

$$\begin{aligned} x_2 &= x_1! \\ x_3 &= (x_1!)! \\ x_5 &= x_1! + 1 \\ x_6 &= (x_1! + 1)! \end{aligned}$$

Proof. It follows from Lemma 2. \square

Theorem 13. *If the equation $x_1! + 1 = x_4^2$ has only finitely many solutions in positive integers, then the statement Ψ_6 guarantees that each such solution (x_1, x_4) belongs to the set $\{(4, 5), (5, 11), (7, 71)\}$.*

Proof. Suppose that the antecedent holds. Let positive integers x_1 and x_4 satisfy $x_1! + 1 = x_4^2$. Then, $x_1, x_4 \in \mathbb{N} \setminus \{0, 1\}$. By Lemma 11, the system C is solvable in positive integers x_2, x_3, x_5, x_6 . Since $C \subseteq B_6$, the statement Ψ_6 implies that $x_6 = (x_1! + 1)! \leq f(6) = f(5)!$. Hence, $x_1! + 1 \leq f(5) = f(4)!$. Consequently, $x_1 < f(4) = 24$. If $x_1 \in \{1, \dots, 23\}$, then $x_1! + 1$ is a square only for $x_1 \in \{4, 5, 7\}$. \square

Theorem 14. *Let \mathcal{X} denote the set of all non-negative integers n which have $((k!)!)!$ digits for some $k \in \{m \in \mathbb{N} : m! + 1 \text{ is a square}\}$. We claim that the statement Ψ_6 implies that \mathcal{X} satisfies conditions (1*) and (2)–(5).*

Proof. Let $d = (((7!)!)!)!$. Since $7! + 1 = 71^2$, we obtain that $\{10^d, \dots, \underbrace{9\dots9}_{d \text{ digits}}\} \subseteq \mathcal{X}$. Hence, $\text{card}(\mathcal{X}) \geq 9 \cdot 10^{d-1}$. By this and Lemma 10, condition (1*) holds. Conditions (2)–(3) and (5) hold trivially. By Theorem 13, the statement Ψ_6 implies that $\underbrace{9\dots9}_{d \text{ digits}}$ is a threshold number of \mathcal{X} . Thus condition (4) holds. \square

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